Approximations and Locally Free Modules

AMeGA & ECI Workshop

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Overview

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- **Part I: Decomposable classes, or the classic jewels**

  1. Some classic examples.

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- **Part II: Deconstructible classes, or the ubiquitous mainstream**
  1. Filtrations and transfinite extensions.
  2. Deconstructibility and approximations.

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- **Part I: Decomposable classes, or the classic jewels**
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- **Part II: Deconstructible classes, or the ubiquitous mainstream**
  1. Filtrations and transfinite extensions.
  2. Deconstructibility and approximations.

- **Part III: Non-deconstructibility, or reaching the limits**
  1. The basic example: Mittag-Leffler modules.
  2. Trees and locally free modules.
  3. Non-deconstructibility via infinite dimensional tilting theory.

Part I: Decomposable classes

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(cutting modules into small blocks)

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Uniformly: $\kappa = \aleph_0$ sufficient for all such R;
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[Faith-Walker'67] **The class $\mathcal{I}_0$ of all injective modules is decomposable, iff R is right noetherian.**
Here, κ depends R; uniqueness by Krull-Schmidt-Azumaya.

Part II: Deconstructible classes

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(blocks put on top of other blocks)

Definition

Let $\mathcal{C} \subseteq \text{Mod-}R$. A module M is $\mathcal{C}$-filtered (or a transfinite extension of the modules in $\mathcal{C}$), provided that there exists an increasing sequence $(M_\alpha \mid \alpha \leq \sigma)$ consisting of submodules of M such that $M_0 = 0$, $M_\sigma = M$,

- $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$, and
- for each $\alpha < \sigma$, $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{C}$.

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Eklof Lemma

The class ${}^\perp\mathcal{C} := \text{KerExt}^1_R(-,\mathcal{C})$ is closed under transfinite extensions for each class of modules $\mathcal{C}$.

In particular, so are the classes $\mathcal{P}_n$ and $\mathcal{F}_n$ of all modules of projective and flat dimension $\leq n$, for each $n < \omega$.

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Definition (Eklof'06)

A class of modules $\mathcal{A}$ is deconstructible, provided there is a cardinal κ such that $\mathcal{A} \subseteq \mathrm{Filt}(\mathcal{A}^{<\kappa})$, where $\mathcal{A}^{<\kappa}$ denotes the class of all $< \kappa$-presented modules from $\mathcal{A}$.

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For each $n < \omega$, the classes $\mathcal{P}_n$ and $\mathcal{F}_n$ are deconstructible.

[Eklof-T.'01], [Šťovíček-T.'09]

For each set of modules $\mathcal{S}$, the class ${}^\perp(\mathcal{S}^\perp)$ is deconstructible.
Here, $\mathcal{S}^\perp := \text{KerExt}^1_R(\mathcal{S}, -)$.

Approximations of modules

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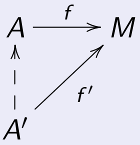
A class of modules $\mathcal{A}$ is precovering if for each module M there is $f \in \mathrm{Hom}_R(A, M)$ with $A \in \mathcal{A}$ such that each $f' \in \mathrm{Hom}_R(A', M)$ with $A' \in \mathcal{A}$ has a factorization through f:



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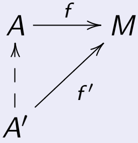

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[Saorín-Šťovíček'11], [Enochs'12]

All deconstructible classes closed under transfinite extensions are precovering.

In particular, so are the classes ${}^{\perp}(\mathcal{S}^{\perp})$ for all sets of modules $\mathcal{S}$.

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2. What about the classes of the form $^{\perp}\mathcal{C}$*?*

Part III: Non-deconstructible classes

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(no block pattern at all)

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[Eklof-Shelah'03]

It is independent of ZFC whether the class of all Whitehead groups ($= {}^{\perp}\{\mathbb{Z}\}$) is precovering (or deconstructible).

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A result in ZFC

A module M is flat Mittag-Leffler provided the functor $M \otimes_R -$ is exact, and for each system of left R-modules $(N_i \mid i \in I)$, the canonical map $M \otimes_R \prod_{i \in I} N_i \to \prod_{i \in I} M \otimes_R N_i$ is monic.

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Assume that R is not right perfect.

- [Herbera-T.'12] The class $\mathcal{FM}$ of all flat Mittag-Leffler modules is closed under transfinite extensions, but it is not deconstructible.
- [Šaroch-T.'12], [Bazzoni-Šťovíček'12] If R is countable, then $\mathcal{FM}$ is not precovering.

Further questions

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Still open

4. Is there a ring R such that the class ${}^{\perp}\mathcal{C}$ is not deconstructible/precovering in ZFC?

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Definition

A module M is locally $\mathcal{F}$-free, if M possesses a subset $\mathcal{S}$ consisting of countably $\mathcal{F}$-filtered modules, such that

- each countable subset of M is contained in an element of $\mathcal{S}$,
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Note: If M is countably generated, then M is locally $\mathcal{F}$-free, iff M is countably $\mathcal{F}$-filtered.

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Theorem (Herbera-T.'12)

Let $\mathcal{F} =$ be the class of all countably presented projective modules. Then the notions of a locally $\mathcal{F}$-free module and a flat Mittag-Leffler module coincide for any ring R.

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For instance, if $R = \mathbb{Z}$, then an abelian group A is flat Mittag-Leffler, iff all countable subgroups of A are free.

In particular, the Baer-Specker group $\mathbb{Z}^\kappa$ is flat Mittag-Leffler for each κ, but not free.

Trees for locally $\mathcal{F}$-free modules

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Let $\mathrm{Br}(T_\kappa)$ denote the set of all branches of T_κ. Each $\nu \in \mathrm{Br}(T_\kappa)$ can be identified with an ω-sequence of ordinals $< \kappa$:

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card $T_\kappa = \kappa$ and card $\mathrm{Br}(T_\kappa) = \kappa^\omega$.

Notation: $\ell(\tau)$ denotes the length of τ for each $\tau \in T_\kappa$.

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$\varinjlim_{\omega} \mathcal{F}$ denotes the class of all Bass modules, i.e., the modules N that are countable direct limits of the modules from $\mathcal{F}$. W.l.o.g., N is the direct limit of a chain

$$F_0 \xrightarrow{g_0} F_1 \xrightarrow{g_1} \dots \xrightarrow{g_{i-1}} F_i \xrightarrow{g_i} F_{i+1} \xrightarrow{g_{i+1}} \dots$$

with $F_i \in \mathcal{F}$ and $g_i \in \mathrm{Hom}_R(F_i, F_{i+1})$ for all $i < \omega$.

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Example

Let $\mathcal{F}$ be the class of all countably generated projective modules. Then the Bass modules coincide with the countably presented flat modules.

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$$\pi_{\nu \restriction i}(x_{\nu i}) = x,$$

$$\pi_{\nu \restriction j}(x_{\nu i}) = g_{j-1} \dots g_i(x) \text{ for each } i < j < \omega,$$

$$\pi_\tau(x_{\nu i}) = 0 \text{ otherwise},$$

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Let $X_{\nu i} := \{x_{\nu i} \mid x \in F_i\}$. Then $X_{\nu i}$ is a submodule of P isomorphic to F_i.

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- L *is locally* $\mathcal{F}$*-free.*

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Lemma (Slávik-T.)

- $\mathcal{L}$ *is closed under transfinite extensions.*
- $\mathcal{L}^\perp \subseteq (\varinjlim_\omega \mathcal{F})^\perp$.

Non-deconstructibility of locally $\mathcal{F}$-free modules

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- $\mathcal{F}$ a class of countably presented modules,
- $\mathcal{L}$ the class of all locally $\mathcal{F}$-free modules,
- $\mathcal{D}$ the class of all direct summands of the modules M that fit into an exact sequence

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where F' is a free module, and C' is countably $\mathcal{F}$-filtered.

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Theorem (Slávik-T.)

Assume there exists a Bass module $N \notin \mathcal{D}$. Then the class $\mathcal{L}$ is not deconstructible.

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Corollary

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Sketch of an alternative proof: If R a non-right perfect ring, then there is a strictly decreasing chain of principal left ideals

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Then there is a non-split pure-exact sequence

$$0 \to R^{(\omega)} \xrightarrow{f} R^{(\omega)} \to N \to 0,$$

where $f(1_i) = 1_i - a_i.1_{i+1}$ for all $i < \omega$. Then $N \notin \mathcal{D} = \mathcal{P}_0$.

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- T has finite projective dimension,
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- there exists an exact sequence $0 \to R \to T_0 \to \cdots \to T_r \to 0$ such that $r < \omega$, and for each $i < r$, $T_i \in \mathrm{Add}(T)$, i.e., T_i is a direct summand of a (possibly infinite) direct sum of copies of T .

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$\mathcal{A}_T := {}^\perp\mathcal{B}_T$ the left tilting class of T.

Some infinite dimensional tilting theory

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Theorem (A model-theoretic characterization of right tilting classes)

Tilting classes are exactly the classes of finite type, i.e., the classes of the form $\mathcal{S}^{\perp}$, where $\mathcal{S}$ is a set of strongly finitely presented modules of bounded projective dimension.

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Let $\mathcal{S}_T := \mathcal{A}_T \cap \text{mod-}R$ and $\bar{\mathcal{A}}_T := \varinjlim \mathcal{S}$. Then $\mathcal{A}_T$ is the class of all direct summands of $\mathcal{S}_T$-filtered modules, and $\mathcal{A}_T \subseteq \bar{\mathcal{A}}_T$.

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Definition

The tilting module T is $\sum$-pure split provided that $\bar{\mathcal{A}}_T = \mathcal{A}_T$, that is, the left tilting class of T is closed under direct limits. Equivalently: Each pure embedding $T_0 \hookrightarrow T_1$ such that $T_0, T_1 \in \text{Add}(T)$ splits.

Some infinite dimensional tilting theory

Theorem (A model-theoretic characterization of right tilting classes)

Tilting classes are exactly the classes of finite type, i.e., the classes of the form $\mathcal{S}^{\perp}$, where $\mathcal{S}$ is a set of strongly finitely presented modules of bounded projective dimension.

Let $\mathcal{S}_T := \mathcal{A}_T \cap \text{mod-}R$ and $\bar{\mathcal{A}}_T := \varinjlim \mathcal{S}$. Then $\mathcal{A}_T$ is the class of all direct summands of $\mathcal{S}_T$-filtered modules, and $\mathcal{A}_T \subseteq \bar{\mathcal{A}}_T$.

Definition

The tilting module T is $\sum$-pure split provided that $\bar{\mathcal{A}}_T = \mathcal{A}_T$, that is, the left tilting class of T is closed under direct limits. Equivalently: Each pure embedding $T_0 \hookrightarrow T_1$ such that $T_0, T_1 \in \text{Add}(T)$ splits.

Example

Let $T = R$. Then T is a tilting module of projective dimension 0, and T is $\sum$-pure split, iff R is a right perfect ring.

Locally free modules and tilting

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The setting

Let R be a countable ring, and T be a non-$\sum$-pure-split tilting module. Let $\mathcal{F}_T$ be the class of all countably presented modules from $\mathcal{A}_T$, and $\mathcal{L}_T$ the class of all locally $\mathcal{F}_T$-free modules.

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Theorem (Slávik-T.)

Assume that $\mathcal{L}_T \subseteq \mathcal{P}_1$, $\mathcal{L}_T$ is closed under direct summands, and $M \in \mathcal{L}_T$ whenever $M \subseteq L \in \mathcal{L}_T$ and $L/M \in \bar{\mathcal{A}}_T$.
Then the class $\mathcal{L}_T$ is not precovering.

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Corollary

If R is countable and non-right perfect, then $\mathcal{FM}$ is not precovering.

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Let R be an indecomposable hereditary artin algebra of infinite representation type, with the Auslander-Reiten translation $\tau := D\mathrm{Ext}^1_R(-, R)$. Then there is a partition of ind-R into three sets:

q ... the indecomposable preinjective modules
(= indecomposable injectives and their τ-shifts),

p ... the indecomposable preprojective modules
(= indecomposable projectives and their τ^--shifts),

t ... the regular modules (the rest).

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Then $p^\perp$ is a right tilting class (and $M \in p^\perp$, iff M has no non-zero direct summands from p).

The tilting module L inducing $p^\perp$ is called the Lukas tilting module.
The left tilting class of L is the class of all Baer modules.

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Let $\mathcal{F}_L$ be the class of all countably presented Baer modules.
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Theorem (Slávik-T.)

If R is a countable indecomposable hereditary artin algebra of infinite representation type, then the class $\mathcal{L}_L$ of all locally Baer modules is not precovering (and hence not deconstructible).