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Einstein metrics on flag manifolds with second Betti number equal to 1

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Situation of the problem -- Einstein manifolds

Dfn. A Riemannian mnfd (M^n, g) is called Einstein if $\mathrm{Ric} = c \cdot g$, for some $c \in \mathbb{R}$.

- A central problem in modern Riemannian geometry is to:

1) Determine which smooth manifolds admit Einstein metrics,

2) Understand the moduli space of these metrics (when they do exist).

The Einstein equation reduces to a system of second order PDE's and general existence results do not exist. Thus new examples are important!

- In dimension 4: ...the story becomes particularly compelling (Hitchin-Thorpe inequality, Kähler-Einstein metrics, gauge theory etc.)
- $\dim \geq 5$: No topological obstructions are known.

Thm. (Hilbert 1915) If M^n is compact $\Rightarrow$ Einstein metrics are critical points of the total scalar curvature functional $T(g) = \int_M \mathrm{Scal}(g) d\mathrm{vol}_g$.

Let $(M = G/K, g)$ be a G-homogeneous Riemannian mnfd.

- **General Problem:** Find G-invariant Einstein metrics on G/K, and classify them up to isometry.

We consider the case $c > 0$ ($c =$ Einstein constant) $\Rightarrow G/K$ is compact & $\pi_1(G/K)$ is finite.

Thm. (Variational approach) **G-invariant Einstein metrics** of volume **1** are explicitly the critical points of $\mathrm{Scal}(g)$ restricted on $\mathcal{M}_1^G$.

Examples in the compact case: Spheres, complex projective spaces, Aloff-Wallach spaces, symmetric spaces of compact type, isotropy irreducible spaces, flag manifolds, Stiefel manifolds, etc.

If $M^n = G/K$ is compact and simply connected $\Rightarrow$

- $\mathbf{n \leq 11}$: M admits **at least 1** G-invariant Einstein metric.

(Böhm-Kerr 05)

- $\mathbf{n \geq 12}$: $\exists$ **counterexamples** : $\mathrm{SU}(4)/\mathrm{SU}(2)$,...

(Wang-Ziller 85, Park-Sakane 93)

• Fix a compact connected semi-simple Lie group G, and let w be an element of the Lie algebra $\mathfrak{g} = T_eG$.

• (Generalized) Flag manifolds $=$ adjoint orbits of G

$$M = \mathrm{Ad}(G)w \subset \mathfrak{g}, \quad (w \in \mathfrak{g}).$$

...in other words: $M = G/K = G/C(S) = G^{\mathbb{C}}/P$.

• Classification : It is based on the Lie theoretic description that admits $M = G/K$ $\Rightarrow$ **painted Dynkin diagrams**.

• Any flag manifold $M = G/K$ admits

- A **finite number** of G-invariant complex structures $\Rightarrow$
- A **finite number** of **G-invariant Kähler-Einstein metrics**

given in terms of the **Koszul form** $\delta_{\mathfrak{m}} = \dfrac{1}{2} \sum_{\alpha \in R_M^+} \alpha$

(Koszul 51, Borel-Hirzebruch 54-56, Alekseevsky-Perelomov 86)

Previous results of homogeneous Einstein metrics

- Compact irreducible Hermitian symmetric spaces $\Rightarrow$ $\exists!$ Kähler-Einstein metric given by $g_B = -$**Killing form**.

- **Full flag manifolds** $M = G/T_{\text{max}}$

Thm. $g_B \in \mathcal{E}(G/T) \Leftrightarrow G \in \{SU(n), SO(2n), E_6, E_7, E_8\}$. (Wang-Ziller 86)

- Classical full flag manifolds $\longrightarrow$ more results are known...

(Alekseevsky 87, Arvanitoyeorgos 91, Sakane 99, Negreiros 06)

Exm. $M = \mathrm{SU}(n)/T_{\text{max}} \Rightarrow \mathcal{E}(M) \leq n!/2 + n + g_B$.

- $\mathrm{SU}(3)/T_{\text{max}}, \mathrm{SU}(4)/T_{\text{max}}, \mathrm{SO}(5)/T_{\text{max}} \Rightarrow$ *full classification*.

- Number of complex homogeneous Einstein metrics (Application of Complex Algebraic geometry, Newton polytopes, etc).

(Graev 06, 07, 12)

Complete classification results/ Real Einstein metrics.

- $\left.\begin{array}{l} M = G/K : \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \\ \Rightarrow \; b_2(M) = 1 \end{array}\right\}$

(Arvanitoyeorgos-Chrysikos 09)

-Method: $\Rightarrow$ variational approach

- $\left.\begin{array}{l} M = G/K : \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \\ \Rightarrow \; b_2(M) = 1 \text{ or } b_2(M) = 2 \end{array}\right\}$

(Kimura 93, Chrysikos-Anastassiou 11)

-Method: $\Rightarrow$ variational approach/normalized Ricci flow.

- $\left.\begin{array}{l} M = G/K : \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \\ \Rightarrow \; b_2(M) = 1 \text{ or } b_2(M) = 2 \end{array}\right\}$

(Arvan-Chrys 10-11)

-Method: $\Rightarrow$ computation of the Ricci tensor & use of the twistor fibration.

Examples with $b_2(G/K) = 2$ (Arvan-Chrys-Sakane 11)

- $G/K = \mathrm{SO}(2\ell)/\mathrm{U}(p) \times \mathrm{U}(\ell - p) \quad (\ell \geq 4,\ 2 \leq p \leq \ell - 2)$
- $G/K = \mathrm{Sp}(\ell)/\mathrm{U}(p) \times \mathrm{U}(\ell - p) \quad (\ell \geq 2,\ 1 \leq p \leq \ell - 1)$

$\Rightarrow \mathcal{E}(G/K)$ is very complicated, since the algebraic system of the homogeneous Einstein equation depends on two parameters ℓ and p.

Thm. $M = \mathrm{SO}(2\ell)/\mathrm{U}(p) \times \mathrm{U}(\ell - p)$ admits exactly **4** non-Kähler $SO(2\ell)$-invariant Einstein metrics for the pairs $(\ell, p) = (12, 6), (10, 5), (8, 4), (7, 4), (7, 3), (6, 4), (6, 3), (6, 2), (5, 3), (5, 2), (4, 2)$, and **2** non-Kähler $\mathrm{SO}(2\ell)$-invariant Einstein metrics for all other cases.

Other results

- $\left.\begin{array}{l} M = G/K = SO(7)/U(1) \times \mathrm{SU}(2) \\ \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5. \end{array}\right\} \Rightarrow \mathcal{E}(G/K) = 5$

(2 Kähler-Einstein and 3 non-Kähler) (Chrysikos 12)

- $\left.\begin{array}{l} G_2/T_{\mathrm{max}} \\ \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6. \end{array}\right\} \Rightarrow \mathcal{E}(G_2/T_{\mathrm{max}}) = 3$

(1 Kähler-Einstein and 2 non-Kähler) (Arvn-Chrys-Sakane 12)

- **Recently:** Classification of invariant Einstein metrics for:

$$\left.\begin{array}{l} M = G/K : \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \\ \& \quad b_2(G/K) = 2. \end{array}\right\}.$$

(Arvn-Chrys-Sakane, submitted)

Flag manifolds $M = G/K$ with second Betti number 1

Fix a compact connected simple Lie group G, $T \subset G$ a maximal torus and let Π be a basis of simple roots w.r.t. T.

- Any flag manifold $M = G/K$ is determined by the choice of a subset $\Pi_K \subset \Pi$.
- Let R, R_K be the root systems of G and K, respectively.

- Roots in $R_M := R \backslash R_K$ are called **complementary roots** $\Rightarrow$ Most important geometric feautures of G/K are given in terms of complementary roots.

Dfn. The **painted Dynkin diagram** associated to $M = G/K$, is obtained from the Dynkin diagram of G by painting black (the nodes of) the simple roots belonging in $\Pi_M := \Pi \backslash \Pi_K$. From this diagram the subsystem Π_K is determined by the subdiagram of white roots and each black node gives rise to one $\mathrm{U}(1)$-component, which their totality forms the center of K.

On the center of K and $\mathfrak{t}$-roots

- We assume that $\Pi_M := \Pi \backslash \Pi_K = \{\alpha_{i_1}, \ldots, \alpha_{i_r}\}$ with $1 \leq i_1 \leq \cdots \leq i_r \leq \ell$. Set

$$\mathfrak{t} = Z(\mathfrak{k}^{\mathbb{C}}) \cap i\mathfrak{h} = \{X \in \mathfrak{h} : \phi(X) = 0 \, \forall \, \phi \in R_K\}. \tag{1}$$

- $Z(\mathfrak{k}^{\mathbb{C}}) = \mathfrak{t} \oplus i\mathfrak{t} = \mathfrak{t}^{\mathbb{C}}$ & $Z(\mathfrak{k}) = i\mathfrak{t}$.
- $\mathfrak{t} \cong \mathfrak{t}^*$ (via the Killing form); A basis of $\mathfrak{t}^*$ is given by the fundamental weights $\{\Lambda_{i_1}, \ldots, \Lambda_{i_r}\} \Rightarrow \dim \mathfrak{t} = |\Pi_M| = r$.

Thm. $\exists$ isomorphism $\hat{\tau} : \mathfrak{t} \to H^2(M; \mathbb{R})$
$\Rightarrow b_2(M) = \dim \mathfrak{t} = r$. (Borel-Hirzebruch 58)

- We consider the linear map $\kappa : \mathfrak{h}^* \to \mathfrak{t}^*, \alpha \mapsto \alpha|_{\mathfrak{t}}$.
- Set $R_{\mathfrak{t}} = \kappa(R) = \kappa(R_M)$ & $\kappa(R_K) = 0$.

Dfn. The elements of $R_{\mathfrak{t}}$ are called $\mathfrak{t}$-roots.

Which is the importance of $\mathfrak{t}$-roots?

- Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a B-orthogonal reductive decomposition of $\mathfrak{g}$.

Thm. There exists a bijective correspondence between $\mathfrak{t}$-roots ξ and irreducible submodules $\mathfrak{m}_\xi$ of the Ad(K)-module $\mathfrak{m}^{\mathbb{C}}$, given by

$$R_{\mathfrak{t}} \ni \xi \ \mapsto \ \mathfrak{m}_\xi = \bigoplus_{\alpha \in R_M : \kappa(\alpha) = \xi} \mathbb{C} E_\alpha .$$

(Siebenthal 64, Alekseevsky 86-87)

- Thus

$$\mathfrak{m}^{\mathbb{C}} = \bigoplus_{\xi \in R_{\mathfrak{t}}} \mathfrak{m}_\xi .$$

- Let $R_{\mathfrak{t}}^{+} = \kappa(R^{+}) = \kappa(R_M^{+})$ be the set of positive $\mathfrak{t}$-roots. Then

$$\mathfrak{m} = \bigoplus_{\xi \in R_{\mathfrak{t}}^{+}} \left(\mathfrak{m}_\xi + \mathfrak{m}_{-\xi}\right)^{\tau} . \tag{2}$$

On the decomposition of the module $\mathfrak{m} \cong T_o G/K$

<u>**Fix notation:**</u> We will call **Dynkin mark** of a <u>simple root</u> $\alpha_j \in \Pi = \{\alpha_1, \ldots, \alpha_\ell\}$, the coefficient $m_j \in \mathbb{Z}^+$ of α_j, appearing in the expression of the **highest root** $\tilde{\alpha} = m_1\alpha_1 + \cdots + m_j\alpha_j + \cdots m_\ell\alpha_\ell \in R^+$ in terms of simple roots.

- The (decomposition of the) isotropy representation of $M = G/K$ depends on the Dynkin marks of the painted black simple roots, i.e., the integers $m_{i_1}, \ldots, m_{i_r}$.
- <u>Assumption</u> From now on we assume that $\Pi_K \subset \Pi$ is such that $\Pi_M = \{\alpha_p\}$ for some **fixed integer** p: $1 \leq p \leq \ell$, i.e. $|\Pi_M| = 1 \Rightarrow b_2(M) = 1$.

Thm. Let $M = G/K$ be a flag manifold defined by $\Pi_M = \{\alpha_p\}$ for some fixed integer p such that $\mathrm{Dyn}(\alpha_p) = N \in \mathbb{Z}^+$. Then, the isotropy representation of M **decomposes into N inequivalent irreducible $\mathrm{Ad}(K)$-submodules**: $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_N$.

- For $k \in \mathbb{Z}$ with $1 \leq k \leq N$, consider the subsets of R^+

$$R^+(\alpha_p, k) = \{\alpha \in R^+ : \alpha = \sum_{j=1}^{\ell} c_j \alpha_j,\ c_p = k\}.$$

- Then $R_M^+ = \bigcup_{1 \leq k \leq N} R^+(\alpha_p, k)$. Consider the subspaces $\mathfrak{m}_k \subset \mathfrak{m}$ defined by

$$\mathfrak{m}_k = \bigoplus_{\alpha \in R^+(\alpha_p, k)} \{\mathbb{R}A_\alpha + \mathbb{R}B_\alpha\}. \tag{3}$$

($A_\alpha = E_\alpha + E_{-\alpha}$ and $B_\alpha = i(E_\alpha - E_{-\alpha})$).

- $\mathfrak{m}_k$ are $\mathrm{Ad}(K)$-invariant submodules of $\mathfrak{m}$ which are inequivalent to each other, of dimension $d_k = \dim_{\mathbb{R}} \mathfrak{m}_k = 2 \cdot |R^+(\alpha_p, k)|$.

$$[\mathfrak{k}, \mathfrak{m}_k] \subset \mathfrak{m}_k, \quad [\mathfrak{m}_k, \mathfrak{m}_k] \subset \mathfrak{h} + \mathfrak{m}_{2k}, \quad [\mathfrak{m}_k, \mathfrak{m}_l] \subset \mathfrak{m}_{k+l} + \mathfrak{m}_{|k-l|} \ (k \neq l).$$

- $\mathfrak{m} = \bigoplus_{k=1}^{N} \mathfrak{m}_k$.
- G-invariant Riemannian metrics on G/K are given by

$$g = x_1 B|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + \ldots + x_N B|_{\mathfrak{m}_N \times \mathfrak{m}_N}, \quad (x_1, \ldots, x_N) \in \mathbb{R}_+^N$$

Basic geometric property of flag manifolds with $b_2 = 1$

Thm. $M = G/K$ admits a unique G-invariant complex structure J defined by an $\mathrm{Ad}(K)$-invariant endomorphism $J_o : \mathfrak{m} \to \mathfrak{m}$ with $J_o^2 = -\mathrm{Id}_{\mathfrak{m}}$, given by

$$J_o E_{\pm\alpha} = \pm i E_{\pm\alpha}, \quad \alpha \in R_M^+.$$

The unique G-invariant Kähler-Einstein metric g_J on G/K compatible with J is given by

$$g_J = B|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + 2 \cdot B|_{\mathfrak{m}_2 \times \mathfrak{m}_2} + \cdots + N \cdot B|_{\mathfrak{m}_N \times \mathfrak{m}_N}.$$

(Borel-Hirzebruch)

Flag manifolds with $b_2(M) = 1$

$M = G/H$ with $b_2(M) = 1$	$\mathcal{E}(M)$
(A) Compact Hermitian Symmetric Spaces	
$\mathrm{SU}(\ell)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(\ell - p))$	$= 1$
$\mathrm{SO}(2\ell + 1)/\mathrm{SO}(2) \times \mathrm{SO}(2\ell - 1)$	$= 1$
$\mathrm{Sp}(\ell)/\mathrm{U}(\ell)$	$= 1$
$\mathrm{SO}(2\ell)/\mathrm{SO}(2) \times \mathrm{SO}(2\ell - 2)$	$= 1$
$\mathrm{SO}(2\ell)/\mathrm{U}(\ell)$	$= 1$
$\mathrm{E}_6/\mathrm{U}(1) \times \mathrm{SO}(10)$	$= 1$
$\mathrm{E}_7/\mathrm{U}(1) \times \mathrm{E}_6$	$= 1$
(B) $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$	
$\mathrm{SO}(2\ell + 1)/\mathrm{U}(\ell - m) \times \mathrm{SO}(2m + 1)$, $(\ell - m \neq 1)$	$= 2$
$\mathrm{Sp}(\ell)/\mathrm{U}(\ell - m) \times \mathrm{Sp}(m)$, $(m \neq 0)$	$= 2$
$\mathrm{SO}(2\ell)/\mathrm{U}(\ell - m) \times \mathrm{SO}(2m)$ $(\ell - m \neq 1,\ m \neq 0)$	$= 2$
$\mathrm{G}_2/\mathrm{U}(2)$, ($\mathrm{U}(2)$ represented by the short root)	$= 2$
$\mathrm{F}_4/\mathrm{SO}(7) \times \mathrm{U}(1)$	$= 2$
$\mathrm{F}_4/\mathrm{Sp}(3) \times \mathrm{U}(1)$	$= 2$
$\mathrm{E}_6/\mathrm{SU}(6) \times \mathrm{U}(1)$	$= 2$
$\mathrm{E}_6/\mathrm{SU}(2) \times \mathrm{SU}(5) \times \mathrm{U}(1)$	$= 2$
$\mathrm{E}_7/\mathrm{SU}(7) \times \mathrm{U}(1)$	$= 2$
$\mathrm{E}_7/\mathrm{SU}(2) \times \mathrm{SO}(10) \times \mathrm{U}(1)$	$= 2$
$\mathrm{E}_7/\mathrm{SO}(12) \times \mathrm{U}(1)$	$= 2$
$\mathrm{E}_8/\mathrm{E}_7 \times \mathrm{U}(1)$	$= 2$
$\mathrm{E}_8/\mathrm{SO}(14) \times \mathrm{U}(1)$	$= 2$

Flag manifolds with $b_2(M) = 1$

$M = G/H$ with $b_2(M) = 1$	$\mathcal{E}(M)$
$(\mathrm{C})\ \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$	
$\mathrm{F}_4\,/\,\mathrm{U}(3) \times \mathrm{SU}(2)$	$= 3$
$\mathrm{E}_6\,/\,\mathrm{U}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(3)$	$= 3$
$\mathrm{E}_7\,/\,\mathrm{U}(3) \times \mathrm{SU}(5)$	$= 3$
$\mathrm{E}_7\,/\,\mathrm{SU}(2) \times \mathrm{SU}(6) \times \mathrm{U}(1)$	$= 3$
$\mathrm{E}_8\,/\,\mathrm{E}_6 \times \mathrm{SU}(2) \times \mathrm{U}(1)$	$= 3$
$\mathrm{E}_8\,/\,\mathrm{U}(8)$	$= 3$
$\mathrm{G}_2\,/\,\mathrm{U}(2)$, $(\mathrm{U}(2)$ represented by the long root$)$	$= 3$
$(\mathrm{D})\ \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4$	
$\mathrm{F}_4\,/\,\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	$= 3$
$\mathrm{E}_7\,/\,\mathrm{SU}(4) \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	$= 3$
$\mathrm{E}_8\,/\,\mathrm{SU}(7) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	$= 3$
$\mathrm{E}_8\,/\,\mathrm{SO}(10) \times \mathrm{SU}(3) \times \mathrm{U}(1)$	$= 5$
$(\mathrm{E})\ \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5$	NEW
$\mathrm{E}_8\,/\,\mathrm{SU}(5) \times \mathrm{SU}(4) \times \mathrm{U}(1)$	$= 6$
$(\mathrm{F})\ \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6$	NEW
$\mathrm{E}_8\,/\,\mathrm{SU}(5) \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$	$= 5$

The **complete classification** *of homogeneous Einstein metrics on flag manifolds with* $b_2 = 1$ - **the new results**

Theorem. A The flag manifold $E_8 / U(1) \times SU(4) \times SU(5)$ admits (up to a scale) precisely **5 non-Kähler** E_8-invariant Einstein metrics, given by: $(x_1 = 1)$

$$
\begin{array}{llll}
(1)\ x_2 \approx 1.0213, & x_3 \approx 0.5460, & x_4 \approx 1.0535, & x_5 \approx 1.1087. \\
(2)\ x_2 \approx 1.0373, & x_3 \approx 1.0471, & x_4 \approx 1.0308, & x_5 \approx 0.2986. \\
(3)\ x_2 \approx 0.5997, & x_3 \approx 1.0837, & x_4 \approx 0.9018, & x_5 \approx 1.2229. \\
(4)\ x_2 \approx 0.7207, & x_3 \approx 1.0254, & x_4 \approx 0.4752, & x_5 \approx 1.0709. \\
(5)\ x_2 \approx 1.0829, & x_3 \approx 1.0408, & x_4 \approx 0.5326, & x_5 \approx 1.1035.
\end{array}
$$

Theorem. B The flag manifold $E_8 / SU(5) \times SU(3) \times SU(2) \times U(1)$ admits (up to a scale) precisely **4 non-Kähler** E_8-invariant Einstein metrics, given by: $(x_1 = 1)$

$$
\begin{array}{lllll}
(1)\ x_2 \approx 0.9548, & x_3 \approx 0.9653, & x_4 \approx 1.0053, & x_5 \approx 0.2900, & x_6 \approx 1.0196. \\
(2)\ x_2 \approx 0.9865, & x_3 \approx 0.6368, & x_4 \approx 1.0685, & x_5 \approx 1.1332, & x_6 \approx 0.9211. \\
(3)\ x_2 \approx 0.9042, & x_3 \approx 0.7782, & x_4 \approx 0.9274, & x_5 \approx 1.0340, & x_6 \approx 0.3599. \\
(4)\ x_2 \approx 0.8230, & x_3 \approx 1.1467, & x_4 \approx 1.1737, & x_5 \approx 1.4266, & x_6 \approx 1.4651.
\end{array}
$$

<u>Thus we conclude that:</u>

Main Theorem. *Every flag manifold G/K of a compact connected simple Lie group G with $b_2(G/K) = 1$, different from an irreducible Hermitian symmetric space of compact type, admits a* ***finite number*** *of* ***non-isometric non-Kähler G-invariant Einstein metrics.***

(Chrysikos-Sakane, submitted)

Finitiness Conjecture Let $M = G/K$ be a compact homogeneous spaces whose isotropy representation decomposes into pairwise inequivalent irreducible submodules, i.e.,

$$\mathfrak{m} = T_o G/K = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s,$$

for some $s \in \mathbb{R}$. Then the set $\mathcal{E}(G/K)$ is **finite**.

(Böhm-Wang-Ziller, 04)

- Let $(M = G/K, \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m})$ be a (compact) homogeneous space and assume that $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ is a B-orthonormal decomposition of $\mathfrak{m}$ into s **pairwise inequivalent** irreducible $\mathrm{Ad}(K)$-submodules. A G-invariant Riemannian metric $g = \langle \ , \ \rangle$ on M has the form

$$g = \langle \ , \ \rangle = x_1 \cdot B|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + \cdots + x_s \cdot B|_{\mathfrak{m}_s \times \mathfrak{m}_s}, \quad (x_1, \ldots, x_s) \in \mathbb{R}^s_+.$$

- Let $\{e_\alpha\}$, $\{e_\beta\}$ and $\{e_\gamma\}$ be B-orthonormal bases of $\mathfrak{m}_i$, $\mathfrak{m}_j$, $\mathfrak{m}_k$ with $1 \leq i, j, k \leq s$, adapted to the decomposition of $\mathfrak{m}$, i.e., $\alpha < \beta$ if $i < j$. We put

$$A^\gamma_{\alpha\beta} = B\left(\left[e_\alpha, e_\beta\right], e_\gamma\right), \quad \text{so that} \quad \left[e_\alpha, e_\beta\right] = \sum_\gamma A^\gamma_{\alpha\beta} e_\gamma.$$

- Set (Wang-Ziller 86)

$$\left[{k \atop ij}\right] = \sum_{\alpha,\beta,\gamma} (A^\gamma_{\alpha\beta})^2. \tag{4}$$

- *The structure constants depend only on the choice of a reductive decomposition!*

- **For flag manifolds:** Let $R_{\mathfrak{t}} \ni \xi_i \leftrightarrow \mathfrak{m}_i$, $R_{\mathfrak{t}} \ni \xi_j \leftrightarrow \mathfrak{m}_j$, $R_{\mathfrak{t}} \ni \xi_k \leftrightarrow \mathfrak{m}_k$.

Dfn. Triples of $\mathfrak{t}$-roots (ξ_i, ξ_j, ξ_k) such that $\xi_i + \xi_j + \xi_k = 0$ are called **symmetric $\mathfrak{t}$-triples**. (Chrysikos 12)

- $\begin{bmatrix} k \\ ij \end{bmatrix} \neq 0$ **if and only if** $\xi_i + \xi_j + \xi_k = 0$.

- The Ricci tensor Ric_g is expressed by an $\mathrm{Ad}(K)$-invariant bilinear form on $\mathfrak{m}$:

$$\mathrm{Ric}_g = x_1 r_1 \cdot B|_{\mathfrak{m}_1} + \cdots + x_s r_s \cdot B|_{\mathfrak{m}_s},$$

where, for any $1 \leq k \leq s$, we have (Park-Sakane 93)

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{i,j} \frac{x_k}{x_i x_j} \begin{bmatrix} k \\ ij \end{bmatrix} - \frac{1}{2d_k} \sum_{i,j} \frac{x_j}{x_k x_i} \begin{bmatrix} j \\ ki \end{bmatrix}. \tag{5}$$

- The **homogeneous Einstein equation** for $g = (x_1, \ldots, x_s) \in \mathbb{R}^s_+$, is the system

$$\{r_1 = c,\ r_2 = c, \ldots, r_s = c\} \Leftrightarrow \{r_1 - r_2 = 0, \ldots, r_{s-1} - r_s = 0\}.$$

Example. **The case $s = 2$, i.e. $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$** $\Rightarrow$ For **flag manifolds**, this is equivalent to say that $\Pi_M = \{\alpha_p : N = 2\}$ where $N := \mathrm{Dyn}(\alpha_p)$.

- Then $R_{\mathfrak{t}}^{+} = \{\overline{\alpha}_p, 2\overline{\alpha}_p\}$ $\Rightarrow$ Only one **symmetric $\mathfrak{t}$-triple** $\overline{\alpha}_p + \overline{\alpha}_p - 2\overline{\alpha}_p = 0$ $\Rightarrow t = \begin{bmatrix} 2 \\ 11 \end{bmatrix} \neq 0$.

- Put $d_1 = \dim \mathfrak{m}_1$ and $d_2 = \dim \mathfrak{m}_2$. We consider G-invariant metrics on G/K given by $g = x_1 \cdot B|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + x_2 \cdot B|_{\mathfrak{m}_2 \times \mathfrak{m}_2}$. Then, the components r_1, r_2 of the Ricci tensor Ric_g are given by

$$\begin{cases} r_1 &= \dfrac{1}{2x_1} - \dfrac{x_2}{2\, d_1\, {x_1}^2}\begin{bmatrix} 2 \\ 11 \end{bmatrix} \\ \\ r_2 &= \dfrac{1}{2x_2} - \dfrac{1}{2\, d_2\, x_2}\begin{bmatrix} 1 \\ 21 \end{bmatrix} + \dfrac{x_2}{4\, d_2\, {x_1}^2}\begin{bmatrix} 2 \\ 11 \end{bmatrix} . \end{cases}$$

• The Einstein equation $r_1 - r_2 = 0 \Leftrightarrow$

$$2td_1x_1^2 - 2d_1d_2x_1^2 - td_1x_2^2 + 2d_1d_2x_1x_2 - 2td_2x_2^2 = 0.$$

• $\exists!$ Kähler-Einstein metric: $g_J = 1 \cdot B|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + 2 \cdot B|_{\mathfrak{m}_2 \times \mathfrak{m}_2}$. Thus we obtain explicitly

$$t = \begin{bmatrix} 2 \\ 11 \end{bmatrix} = \frac{d_1 d_2}{d_1 + 4d_2}.$$

Thm. Let $M = G/K$ be a generalized flag manifold with two isotropy summands. Then M admits precisely **1 non-Kähler** G-invariant Einstein metric, given by

$$x_1 = 1, \qquad x_2 = \frac{4d_2}{d_1 + 2d_2}.$$

Methods for computing the non-zero structure constants.

Let $\mathcal{S}$ be set of non-zero $\begin{bmatrix} k \\ ij \end{bmatrix}$ with respect the decomposition $\mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_N$.

- $\mathfrak{m}$ isotropy irreducible $\Rightarrow \mathcal{S} = \emptyset$
- $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \Rightarrow |\mathcal{S}| = 1$
- $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \Rightarrow |\mathcal{S}| = 2$
- $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_4 \Rightarrow |\mathcal{S}| = 4$
- $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \Rightarrow |\mathcal{S}| = 6$
- $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \oplus \mathfrak{m}_6 \Rightarrow |\mathcal{S}| = 9$

- $N = 2 \Rightarrow \{r_1 - r_2 = 0\} \Rightarrow$ Kähler-Einstein metric.
- $N = 3 \Rightarrow \{r_1 - r_2 = 0, r_2 - r_3 = 0\} \Rightarrow$ Kähler-Einstein metric.
- $N = 4 \Rightarrow \{r_1 - r_2 = 0, \ldots, r_3 - r_4 = 0\} \Rightarrow$ Kähler-Einstein metric & twistor fibration.
- $N = 5 \Rightarrow \{r_1 - r_2 = 0, \ldots, r_4 - r_5 = 0\} \Rightarrow$ Kähler-Einstein metric & method of Riemannian submersions
- $N = 6 \Rightarrow \{r_1 - r_2 = 0, \ldots, r_5 - r_6 = 0\} \Rightarrow$ Kähler-Einstein metric & method of Riemannian submersions & twistor fibration.

Which is the **philoshophy** of the method of Riemannian submersions **here?**

Notice that: the homogeneous space G/K and the Lie group G have **in common some non-zero structure constants**, with respect a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. We assume

$$
\begin{aligned}
\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} &= \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_p \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \cdots \oplus \mathfrak{m}_N \\
&= \underbrace{\mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_p}_{\mathfrak{k}} \oplus \underbrace{\mathfrak{m}_{p+1} \oplus \cdots \oplus \mathfrak{m}_{p+N}}_{\mathfrak{m}} .
\end{aligned}
$$

Left-invariant metrics on G	G-invariant metrics on G/K
$\langle\ ,\ \rangle = \sum_{i=0}^{p} u_i \cdot B\vert_{\mathfrak{k}_i \times \mathfrak{k}_i} + \sum_{j=1}^{N} x_j \cdot B\vert_{\mathfrak{m}_j \times \mathfrak{m}_j}$ $= \sum_{i=0}^{p+N} y_i \cdot B\vert_{\mathfrak{m}_i \times \mathfrak{m}_i}$ with $y_i \in \mathbb{R}^+$	$(\ ,\ ) = \sum_{j=p+1}^{p+N} x_j \cdot B\vert_{\mathfrak{m}_j \times \mathfrak{m}_j}$ with $x_j \in \mathbb{R}^+$
Ricci tenosr Ric of G	Ricci tensor $\overline{\text{Ric}}$ of G/K
$\text{Ric} = \sum_{i=0}^{p+N} y_i \cdot r_i \cdot B\vert_{\mathfrak{m}_i \times \mathfrak{m}_i}$	$\overline{\text{Ric}} = \sum_{j=p+1}^{p+N} x_j \cdot \overline{r}_j \cdot B\vert_{\mathfrak{m}_j \times \mathfrak{m}_j}$

Thm. *(1)* The components $r_0, r_1, \cdots, r_{p+N}$ of the Ricci tensor Ric of the metric $\langle\ ,\ \rangle$ on G are given by

$$r_k = \frac{1}{2y_k} + \frac{1}{4d_k}\sum_{j,i}\frac{y_k}{y_j y_i}\begin{bmatrix}k\\ ji\end{bmatrix} - \frac{1}{2d_k}\sum_{j,i}\frac{y_j}{y_k y_i}\begin{bmatrix}j\\ ki\end{bmatrix}, \tag{6}$$

for $0 \leq k \leq p+N$. Here the sum is taken over all $i, j = 0, 1, \cdots, p+N$. Moreover, for each k we have $\sum_{i,j}\begin{bmatrix}j\\ ki\end{bmatrix} = d_k$.

(2) The components $\bar{r}_{p+1}, \cdots, \bar{r}_{p+N}$ of the Ricci tensor $\overline{\text{Ric}}$ on G/K are given by

$$\bar{r}_k = \frac{1}{2x_k} + \frac{1}{4d_k}\sum_{j,i}\frac{x_k}{x_j x_i}\begin{bmatrix}k\\ ji\end{bmatrix} - \frac{1}{2d_k}\sum_{j,i}\frac{x_j}{x_k x_i}\begin{bmatrix}j\\ ki\end{bmatrix}, \tag{7}$$

for $p+1 \leq k \leq p+N$. Here the sum is taken over all $i, j = p+1, \cdots, p+N$.

• Construct now a **new reductive decomposition** $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ with $\mathfrak{k} \subset \mathfrak{h}$ $\Rightarrow$ we get a fibration $G/K \to G/H$ with fiber H/K and $\mathfrak{p} = T_o G/H$. Assume that

$$\begin{aligned} \mathfrak{g} &= \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_q \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \cdots \oplus \mathfrak{p}_t \\ &= \underbrace{\mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_q}_{\mathfrak{h}} \oplus \underbrace{\mathfrak{p}_{q+1} \oplus \cdots \oplus \mathfrak{p}_{q+t}}_{\mathfrak{p}} . \end{aligned}$$

• We consider **new left-invariant metrics** in G , given by

$$\langle\langle\ ,\ \rangle\rangle = \sum_{j=1}^{q+t} y'_j \cdot B|_{\mathfrak{p}_j \times \mathfrak{p}_j}, \quad \text{for some **new positive real parameters** } y'_j \in \mathbb{R}^+.$$

• Let $\widetilde{\mathrm{Ric}} = \sum_{j=1}^{q+t} \widetilde{r_j} \cdot y'_j \cdot B|_{\mathfrak{p}_j \times \mathfrak{p}_j}$ be the Ricci tensor of $\langle\langle\ ,\ \rangle\rangle$.

• The left-invariant metrics $\langle\ ,\ \rangle$ and $\langle\langle\ ,\ \rangle\rangle$ must coincide for some appropriate choice of the parameters y_i and y'_j. For this values, $\mathrm{Ric} = \widetilde{\mathrm{Ric}}$ $\Rightarrow$ we get **valuable** relations between the non-zero structure constants $\begin{bmatrix} k \\ ji \end{bmatrix}$ of G/K.

The flag manifold $M = G/K = \mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(4) \times \mathrm{SU}(5)$

- It is obtained by choosing $\Pi_K = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$, i.e., $\Pi_M = \{\alpha_4\}$. A reductive decomposition of $\mathfrak{g} = \mathfrak{e}_8$ (w.r.t. B), is given by

$$\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_4 \oplus \mathfrak{m}_5 \tag{8}$$

where $\mathfrak{k}_0 = Z(\mathfrak{k}) = \mathfrak{u}(1)$ and $\mathfrak{k}_1 = \mathfrak{su}(4)$, $\mathfrak{k}_2 = \mathfrak{su}(5)$.
- Notice that:

$$\begin{array}{llll} d_0 = \dim \mathfrak{k}_0 = 1, & d_2 = \dim \mathfrak{k}_2 = 24 & d_4 = \dim \mathfrak{m}_2 = 60 & d_6 = \dim \mathfrak{m}_4 = 20 \\ d_1 = \dim \mathfrak{k}_1 = 15 & d_3 = \dim \mathfrak{m}_1 = 80 & d_5 = \dim \mathfrak{m}_3 = 40 & d_7 = \dim \mathfrak{m}_5 = 8 \end{array}$$

- We consider left-invariant Riemannian metrics on the compact Lie group E_8, given by

$$\begin{aligned} \langle \ , \ \rangle \ &= \ u_0 \cdot B|_{\mathfrak{k}_0} + u_1 \cdot B|_{\mathfrak{k}_1} + u_2 \cdot B|_{\mathfrak{k}_2} \\ &+ \ x_1 \cdot B|_{\mathfrak{m}_1} + x_2 \cdot B|_{\mathfrak{m}_2} + x_3 \cdot B|_{\mathfrak{m}_3} + x_4 \cdot B|_{\mathfrak{m}_4} + x_5 \cdot B|_{\mathfrak{m}_5}, \end{aligned}$$

for positive real numbers $u_0, u_1, u_2, x_1, x_2, x_3, x_4, x_5$. Note that the left-invariant metric is also $\mathrm{Ad}(K)$-invariant.

- The **non-zero** structure constants $\begin{bmatrix} k \\ ij \end{bmatrix}$ of $(G, \langle\ ,\ \rangle)$ w.r.t. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, are the following (and their symmetries):

$$\begin{bmatrix} 3 \\ 03 \end{bmatrix}, \begin{bmatrix} 4 \\ 04 \end{bmatrix}, \begin{bmatrix} 5 \\ 05 \end{bmatrix}, \begin{bmatrix} 6 \\ 06 \end{bmatrix}, \begin{bmatrix} 7 \\ 07 \end{bmatrix}, \begin{bmatrix} 1 \\ 11 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \end{bmatrix}, \begin{bmatrix} 4 \\ 14 \end{bmatrix}, \begin{bmatrix} 5 \\ 15 \end{bmatrix}, \begin{bmatrix} 6 \\ 16 \end{bmatrix}, \begin{bmatrix} 7 \\ 17 \end{bmatrix},$$

$$\begin{bmatrix} 2 \\ 22 \end{bmatrix}, \begin{bmatrix} 3 \\ 23 \end{bmatrix}, \begin{bmatrix} 4 \\ 24 \end{bmatrix}, \begin{bmatrix} 5 \\ 25 \end{bmatrix}, \begin{bmatrix} 6 \\ 26 \end{bmatrix}, \begin{bmatrix} 4 \\ 33 \end{bmatrix}, \begin{bmatrix} 5 \\ 34 \end{bmatrix}, \begin{bmatrix} 6 \\ 35 \end{bmatrix}, \begin{bmatrix} 7 \\ 36 \end{bmatrix}, \begin{bmatrix} 6 \\ 44 \end{bmatrix}, \begin{bmatrix} 7 \\ 45 \end{bmatrix}.$$

- G-invariant Riemannian metric on G/K are given by

$$(\ ,\ ) = x_1 \cdot B|_{\mathfrak{m}_1} + x_2 \cdot B|_{\mathfrak{m}_2} + x_3 \cdot B|_{\mathfrak{m}_3} + x_4 \cdot B|_{\mathfrak{m}_4} + x_5 \cdot B|_{\mathfrak{m}_5}.$$

- The **non-zero** structure constants $\begin{bmatrix} k \\ ij \end{bmatrix}$ of $(G/K, (\ ,\ ))$ w.r.t. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, are the triples (and their symmetries)

$$\begin{bmatrix} 4 \\ 33 \end{bmatrix}, \begin{bmatrix} 5 \\ 34 \end{bmatrix}, \begin{bmatrix} 6 \\ 35 \end{bmatrix}, \begin{bmatrix} 7 \\ 36 \end{bmatrix}, \begin{bmatrix} 6 \\ 44 \end{bmatrix}, \begin{bmatrix} 7 \\ 45 \end{bmatrix}.$$

- The componenets r_i of the Ricci tensor Ric of $\langle\ ,\ \rangle$ on G are given by:

$$
\begin{aligned}
r_0 &= \frac{u_0}{4\,x_1{}^2}\genfrac{[}{]}{0pt}{}{0}{33} + \frac{u_0}{4\,x_2{}^2}\genfrac{[}{]}{0pt}{}{0}{44} + \frac{u_0}{4\,x_3{}^2}\genfrac{[}{]}{0pt}{}{0}{55} + \frac{u_0}{4\,x_4{}^2}\genfrac{[}{]}{0pt}{}{0}{66} + \frac{u_0}{4\,x_5{}^2}\genfrac{[}{]}{0pt}{}{0}{77} \\
r_1 &= \frac{1}{4\,d_1\,u_1}\genfrac{[}{]}{0pt}{}{1}{11} + \frac{u_1}{4\,d_1\,x_1{}^2}\genfrac{[}{]}{0pt}{}{1}{33} + \frac{u_1}{4\,d_1\,x_2{}^2}\genfrac{[}{]}{0pt}{}{1}{44} + \frac{u_1}{4\,d_1\,x_3{}^2}\genfrac{[}{]}{0pt}{}{1}{55} \\
&+ \frac{u_1}{4\,d_1\,x_4{}^2}\genfrac{[}{]}{0pt}{}{1}{66} + \frac{u_1}{4\,d_1\,x_5{}^2}\genfrac{[}{]}{0pt}{}{1}{77} \\
r_2 &= \frac{1}{4\,d_2\,u_2}\genfrac{[}{]}{0pt}{}{2}{22} + \frac{u_2}{4\,d_2\,x_1{}^2}\genfrac{[}{]}{0pt}{}{2}{33} + \frac{u_2}{4\,d_1\,x_2{}^2}\genfrac{[}{]}{0pt}{}{2}{44} + \frac{u_2}{4\,d_1\,x_3{}^2}\genfrac{[}{]}{0pt}{}{2}{55} \\
&+ \frac{u_2}{4\,d_1\,x_4{}^2}\genfrac{[}{]}{0pt}{}{2}{66} \\
r_3 &= \frac{1}{2x_1} - \frac{1}{2\,d_3}\genfrac{[}{]}{0pt}{}{4}{33}\frac{x_2}{x_1{}^2} + \frac{1}{2\,d_3}\genfrac{[}{]}{0pt}{}{3}{45}\left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\right) \\
&+ \frac{1}{2\,d_3}\genfrac{[}{]}{0pt}{}{3}{56}\left(\frac{x_1}{x_3x_4} - \frac{x_3}{x_1x_4} - \frac{x_4}{x_1x_3}\right) + \frac{1}{2\,d_3}\genfrac{[}{]}{0pt}{}{3}{67}\left(\frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5} - \frac{x_5}{x_1x_4}\right) \\
&- \frac{1}{2\,d_3\,x_1{}^2}\left(u_0\genfrac{[}{]}{0pt}{}{0}{33} + u_1\genfrac{[}{]}{0pt}{}{1}{33} + u_2\genfrac{[}{]}{0pt}{}{2}{33} \right)
\end{aligned}
$$

$$
\begin{aligned}
r_4 &= \frac{1}{2x_2} + \frac{1}{4\,d_4}\genfrac{[}{]}{0pt}{}{4}{33}\left(\frac{x_2}{{x_1}^2} - \frac{2}{x_2}\right) - \frac{1}{2\,d_4}\genfrac{[}{]}{0pt}{}{6}{44}\frac{x_4}{{x_2}^2} \\
&+ \frac{1}{2\,d_4}\genfrac{[}{]}{0pt}{}{4}{35}\left(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_2x_1}\right) + \frac{1}{2\,d_4}\genfrac{[}{]}{0pt}{}{4}{57}\left(\frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5} - \frac{x_5}{x_2x_3}\right) \\
&- \frac{1}{2\,d_4\,{x_2}^2}\left(u_0\genfrac{[}{]}{0pt}{}{0}{44} + u_1\genfrac{[}{]}{0pt}{}{1}{44} + u_2\genfrac{[}{]}{0pt}{}{2}{44} \right)
\end{aligned}
$$

$$
\begin{aligned}
r_5 &= \frac{1}{2x_3} + \frac{1}{2\,d_5}\genfrac{[}{]}{0pt}{}{5}{34}\left(\frac{x_3}{x_1x_2} - \frac{x_2}{x_3x_1} - \frac{x_1}{x_3x_2}\right) + \frac{1}{2\,d_5}\genfrac{[}{]}{0pt}{}{5}{36}\left(\frac{x_3}{x_1x_4} - \frac{x_1}{x_3x_4} - \frac{x_4}{x_1x_3}\right) \\
&+ \frac{1}{2\,d_5}\genfrac{[}{]}{0pt}{}{5}{47}\left(\frac{x_3}{x_2x_5} - \frac{x_2}{x_3x_5} - \frac{x_5}{x_3x_2}\right) - \frac{1}{2\,d_5\,{x_3}^2}\left(u_0\genfrac{[}{]}{0pt}{}{0}{55} + u_1\genfrac{[}{]}{0pt}{}{1}{55} + u_2\genfrac{[}{]}{0pt}{}{2}{55} \right)
\end{aligned}
$$

$$
\begin{aligned}
r_6 &= \frac{1}{2x_4} + \frac{1}{4\,d_6}\genfrac{[}{]}{0pt}{}{6}{44}\left(\frac{x_4}{{x_2}^2} - \frac{2}{x_4}\right) + \frac{1}{2\,d_6}\genfrac{[}{]}{0pt}{}{6}{35}\left(\frac{x_4}{x_1x_3} - \frac{x_1}{x_3x_4} - \frac{x_3}{x_4x_1}\right) \\
&+ \frac{1}{2\,d_6}\genfrac{[}{]}{0pt}{}{6}{37}\left(\frac{x_4}{x_1x_5} - \frac{x_1}{x_4x_5} - \frac{x_5}{x_1x_4}\right) - \frac{1}{2\,d_6\,{x_4}^2}\left(u_0\genfrac{[}{]}{0pt}{}{0}{66} + u_1\genfrac{[}{]}{0pt}{}{1}{66} + u_2\genfrac{[}{]}{0pt}{}{2}{66} \right)
\end{aligned}
$$

$$
\begin{aligned}
r_7 &= \frac{1}{2x_5} + \frac{1}{2\,d_7}\genfrac{[}{]}{0pt}{}{7}{45}\left(\frac{x_5}{x_2x_3} - \frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5}\right) \\
&+ \frac{1}{2\,d_7}\genfrac{[}{]}{0pt}{}{7}{36}\left(\frac{x_5}{x_1x_4} - \frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5}\right) - \frac{1}{2\,d_7\,{x_5}^2}\left(u_0\genfrac{[}{]}{0pt}{}{0}{77} + u_1\genfrac{[}{]}{0pt}{}{1}{77} \right).
\end{aligned}
$$

●The componenets $\overline{r}_i$ of the Ricci tensor $\overline{\mathrm{Ric}}$ of $(\ ,\ )$ on G/K are given by:

$$
\begin{aligned}
\bar{r}_1 &= \frac{1}{2x_1} - \frac{1}{2\,d_3}\begin{bmatrix}4\\33\end{bmatrix}\frac{x_2}{{x_1}^2} + \frac{1}{2\,d_3}\begin{bmatrix}3\\45\end{bmatrix}\left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\right)\\
&+\frac{1}{2\,d_3}\begin{bmatrix}3\\56\end{bmatrix}\left(\frac{x_1}{x_3x_4} - \frac{x_3}{x_1x_4} - \frac{x_4}{x_1x_3}\right) + \frac{1}{2\,d_3}\begin{bmatrix}3\\67\end{bmatrix}\left(\frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5} - \frac{x_5}{x_1x_4}\right)\\
\bar{r}_2 &= \frac{1}{2x_2} + \frac{1}{4\,d_4}\begin{bmatrix}4\\33\end{bmatrix}\left(\frac{x_2}{{x_1}^2} - \frac{2}{x_2}\right) - \frac{1}{2\,d_4}\begin{bmatrix}6\\44\end{bmatrix}\frac{x_4}{{x_2}^2}\\
&+\frac{1}{2\,d_4}\begin{bmatrix}4\\35\end{bmatrix}\left(\frac{x_2}{x_1x_3} - \frac{x_1}{x_2x_3} - \frac{x_3}{x_2x_1}\right)+\frac{1}{2\,d_4}\begin{bmatrix}4\\57\end{bmatrix}\left(\frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5} - \frac{x_5}{x_2x_3}\right)\\
\bar{r}_3 &= \frac{1}{2x_3}+\frac{1}{2\,d_5}\begin{bmatrix}5\\34\end{bmatrix}\left(\frac{x_3}{x_1x_2} - \frac{x_2}{x_3x_1} - \frac{x_1}{x_3x_2}\right) + \frac{1}{2\,d_5}\begin{bmatrix}5\\36\end{bmatrix}\left(\frac{x_3}{x_1x_4} - \frac{x_1}{x_3x_4} - \frac{x_4}{x_1x_3}\right)\\
&+\frac{1}{2\,d_5}\begin{bmatrix}5\\47\end{bmatrix}\left(\frac{x_3}{x_2x_5} - \frac{x_2}{x_3x_5} - \frac{x_5}{x_3x_2}\right)\\
\bar{r}_4 &= \frac{1}{2x_4} + \frac{1}{4\,d_6}\begin{bmatrix}6\\44\end{bmatrix}\left(\frac{x_4}{{x_2}^2} - \frac{2}{x_4}\right)+\frac{1}{2\,d_6}\begin{bmatrix}6\\35\end{bmatrix}\left(\frac{x_4}{x_1x_3} - \frac{x_1}{x_3x_4} - \frac{x_3}{x_4x_1}\right)\\
&+\frac{1}{2\,d_6}\begin{bmatrix}6\\37\end{bmatrix}\left(\frac{x_4}{x_1x_5} - \frac{x_1}{x_4x_5} - \frac{x_5}{x_1x_4}\right)\\
\bar{r}_5 &= \frac{1}{2x_5} + \frac{1}{2\,d_7}\begin{bmatrix}7\\45\end{bmatrix}\left(\frac{x_5}{x_2x_3} - \frac{x_2}{x_3x_5} - \frac{x_3}{x_2x_5}\right)\\
&+\frac{1}{2\,d_7}\begin{bmatrix}7\\36\end{bmatrix}\left(\frac{x_5}{x_1x_4} - \frac{x_1}{x_4x_5} - \frac{x_4}{x_1x_5}\right).
\end{aligned}
$$

The computation of the non-zero structure constants for $(G/K, \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m})$.

(A) *Use of the known Kähler-Einstein metric.* We substitute the values $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5$ in the system $\{\bar{r}_1 = \bar{r}_2 = \bar{r}_3 = \bar{r}_4 = \bar{r}_5\}$:

$$
\begin{aligned}
\frac{1}{2} - \frac{1}{d_3}\left(\begin{bmatrix}4\\33\end{bmatrix} + \begin{bmatrix}5\\34\end{bmatrix} + \begin{bmatrix}6\\35\end{bmatrix} + \begin{bmatrix}7\\36\end{bmatrix}\right) &= \frac{1}{4} + \frac{1}{d_4}\left(\frac{1}{4}\begin{bmatrix}4\\33\end{bmatrix} - \frac{1}{2}\begin{bmatrix}5\\34\end{bmatrix} - \frac{1}{2}\begin{bmatrix}6\\44\end{bmatrix} - \frac{1}{2}\begin{bmatrix}7\\45\end{bmatrix}\right) \\
= \frac{1}{6} + \frac{1}{d_5}\left(\frac{1}{3}\begin{bmatrix}5\\34\end{bmatrix} - \frac{1}{3}\begin{bmatrix}6\\35\end{bmatrix} - \frac{1}{3}\begin{bmatrix}7\\45\end{bmatrix}\right) &= \frac{1}{8} + \frac{1}{d_6}\left(\frac{1}{4}\begin{bmatrix}6\\35\end{bmatrix} - \frac{1}{4}\begin{bmatrix}7\\36\end{bmatrix} + \frac{1}{8}\begin{bmatrix}6\\44\end{bmatrix}\right) \\
&= \frac{1}{10} + \frac{1}{d_7}\left(\frac{1}{5}\begin{bmatrix}7\\36\end{bmatrix} + \frac{1}{5}\begin{bmatrix}7\\45\end{bmatrix}\right).
\end{aligned}
$$

$\Rightarrow$ We obtain a system of **4** equations with **6** unknowns, namely the triples

$$
\begin{bmatrix}4\\33\end{bmatrix}, \quad \begin{bmatrix}5\\34\end{bmatrix}, \quad \begin{bmatrix}6\\35\end{bmatrix}, \quad \begin{bmatrix}7\\36\end{bmatrix}, \quad \begin{bmatrix}6\\44\end{bmatrix}, \quad \begin{bmatrix}7\\45\end{bmatrix}.
$$

- In order to compute them explicitly, we need two more equations.

(B) *Construction of a new reductive decomposition.*

- We put

$$\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{m}_5, \quad \mathfrak{h}_1 = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \mathfrak{m}_5, \quad \mathfrak{p}_1 = \mathfrak{m}_1 \oplus \mathfrak{m}_4, \quad \mathfrak{p}_2 = \mathfrak{m}_2 \oplus \mathfrak{m}_3.$$

- Then $\mathfrak{h} \subset \mathfrak{g}$ and $\mathfrak{h}_1 \subset \mathfrak{g}$. In particular it is $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{k}_2$, where $\mathfrak{k}_2 = \mathfrak{su}(5)$, and for dimensional reasons we also obtain that $\mathfrak{h}_1 = \mathfrak{su}(5)$.
- In this way we define a reductive decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} = (\mathfrak{h}_1 \oplus \mathfrak{k}_2) \oplus (\mathfrak{p}_1 \oplus \mathfrak{p}_2).$$

- Since $\mathfrak{k} \subset \mathfrak{h} \Rightarrow$ there is a natural fibration $G/K \to G/H$, given by

$$\mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(4) \times \mathrm{SU}(5) \to \mathrm{E}_8 / \mathrm{SU}(5) \times \mathrm{SU}(5).$$

- The base space G/H has two isotropy summands, namely $\mathfrak{p} = T_oG/H = \mathfrak{p}_1 \oplus \mathfrak{p}_2$.

(Wang-Ziller 85)

• We consider new left-invariant metrics on $G = \mathrm{E}_8$ which are also $\mathrm{Ad}(H)$-invariant:

$$\langle\langle\ ,\ \rangle\rangle = y_1 \cdot B|_{\mathfrak{h}_1} + y_2 \cdot B|_{\mathfrak{k}_2} + y_3 \cdot B|_{\mathfrak{p}_1} + y_4 \cdot B|_{\mathfrak{p}_2}, \quad (y_1, y_2, y_3, y_4) \in \mathbb{R}^4_+. \tag{9}$$

• Notice that the $\mathrm{Ad}(H)$-modules $\mathfrak{p}_1, \mathfrak{p}_2$ are satisfying:

$$[\mathfrak{p}_1,\ \mathfrak{p}_1] \subset \mathfrak{p}_2 \oplus \mathfrak{h}, \quad [\mathfrak{p}_1,\ \mathfrak{p}_2] \subset \mathfrak{p}_1 \oplus \mathfrak{p}_2, \quad [\mathfrak{p}_2,\ \mathfrak{p}_2] \subset \mathfrak{p}_1 \oplus \mathfrak{h}. \tag{10}$$

•Thus, for $(\mathrm{E}_8, \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}, \langle\langle\ ,\ \rangle\rangle)$, the **new non-zero structure constants** are the following (and their symmetries):

$$\begin{bmatrix} 1 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 13 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 14 \end{bmatrix}, \begin{bmatrix} 2 \\ 22 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 23 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 24 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 33 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 34 \end{bmatrix}.$$

- The components $\widetilde{r}_i$ of the Ricci tensor $\widetilde{\mathrm{Ric}}$ for the left-invariant metric $\langle\langle\ ,\ \rangle\rangle$ on E_8 are of the form:

$$\begin{cases} \widetilde{r}_1 = \dfrac{1}{4\, f_1\, y_1} \left[\begin{bmatrix} 1 \\ 11 \end{bmatrix}\right] + \dfrac{y_1}{4\, f_1\, y_3{}^2} \left[\begin{bmatrix} 1 \\ 33 \end{bmatrix}\right] + \dfrac{y_1}{4\, f_1\, y_4{}^2} \left[\begin{bmatrix} 1 \\ 44 \end{bmatrix}\right] \\ \widetilde{r}_2 = \dfrac{1}{4\, f_2\, y_2} \left[\begin{bmatrix} 2 \\ 22 \end{bmatrix}\right] + \dfrac{y_2}{4\, f_2\, y_3{}^2} \left[\begin{bmatrix} 2 \\ 33 \end{bmatrix}\right] + \dfrac{y_2}{4\, f_2\, y_4{}^2} \left[\begin{bmatrix} 2 \\ 44 \end{bmatrix}\right] \\ \widetilde{r}_3 = \dfrac{1}{2y_3} + \dfrac{y_3}{4\, f_3\, y_4{}^2} \left[\begin{bmatrix} 3 \\ 44 \end{bmatrix}\right] - \dfrac{1}{2\, f_3} \left(\dfrac{y_1}{y_3{}^2} \left[\begin{bmatrix} 1 \\ 33 \end{bmatrix}\right] + \dfrac{y_2}{y_3{}^2} \left[\begin{bmatrix} 2 \\ 33 \end{bmatrix}\right] + \dfrac{y_4}{y_3{}^2} \left[\begin{bmatrix} 4 \\ 33 \end{bmatrix}\right] + \dfrac{1}{y_3} \left[\begin{bmatrix} 4 \\ 34 \end{bmatrix}\right] \right) \\ \widetilde{r}_4 = \dfrac{1}{2y_4} + \dfrac{y_4}{4\, f_4\, y_3{}^2} \left[\begin{bmatrix} 4 \\ 33 \end{bmatrix}\right] - \dfrac{1}{2\, f_4} \left(\dfrac{y_1}{y_4{}^2} \left[\begin{bmatrix} 1 \\ 44 \end{bmatrix}\right] + \dfrac{y_2}{y_4{}^2} \left[\begin{bmatrix} 2 \\ 44 \end{bmatrix}\right] + \dfrac{y_3}{y_4{}^2} \left[\begin{bmatrix} 3 \\ 44 \end{bmatrix}\right] + \dfrac{1}{y_4} \left[\begin{bmatrix} 3 \\ 34 \end{bmatrix}\right] \right). \end{cases}$$

- Here it is $f_1 = \dim \mathfrak{h}_1 = 24$, $f_2 = \dim \mathfrak{k}_2 = 24$, $f_3 = \dim \mathfrak{p}_1 = 100$ and $f_4 = \dim \mathfrak{p}_2 = 100$.

- Observe that the metrics $\langle\ ,\ \rangle$ and $\langle\langle\ ,\ \rangle\rangle$ coincide if we set

$$y_1 = u_0 = u_1 = x_5, \quad y_2 = u_2, \quad y_3 = x_1 = x_4, \quad y_4 = x_2 = x_3.$$

- For these values the components of the Ricci tensors $\widetilde{\mathrm{Ric}}$ and Ric must be equal.

In particular:

- From $y_3 = x_1 = x_4 \Rightarrow \tilde{r}_3 = r_3 = r_6$,
- From $y_4 = x_2 = x_3 \Rightarrow \tilde{r}_4 = r_4 = r_5$.

- By using the first relation, we obtain the following equations:

$$\frac{1}{2\, f_3}\begin{bmatrix} 4 \\ 34 \end{bmatrix} = \frac{1}{d_3}\begin{bmatrix} 5 \\ 34 \end{bmatrix} = \frac{1}{2d_6}\begin{bmatrix} 6 \\ 44 \end{bmatrix},$$

$$\frac{1}{2\, f_3}\begin{bmatrix} 4 \\ 33 \end{bmatrix} = \frac{1}{2d_3}\begin{bmatrix} 4 \\ 33 \end{bmatrix} + \frac{1}{2d_3}\begin{bmatrix} 6 \\ 35 \end{bmatrix} = \frac{1}{2d_6}\begin{bmatrix} 6 \\ 35 \end{bmatrix}$$

- In this way we obtain a system of **6** equations & **6** unknowns:

$$
\begin{aligned}
60 - 4\begin{bmatrix}4\\33\end{bmatrix} - \begin{bmatrix}5\\34\end{bmatrix} - 3\begin{bmatrix}6\\35\end{bmatrix} - 3\begin{bmatrix}7\\36\end{bmatrix} + 2\begin{bmatrix}6\\44\end{bmatrix} + \begin{bmatrix}7\\45\end{bmatrix} &= 0,\\
4 + 2\begin{bmatrix}6\\35\end{bmatrix} - 6\begin{bmatrix}7\\36\end{bmatrix} + \begin{bmatrix}6\\44\end{bmatrix} - 4\begin{bmatrix}7\\45\end{bmatrix} &= 0,\\
20 + \begin{bmatrix}4\\33\end{bmatrix} - 4\begin{bmatrix}5\\34\end{bmatrix} + 2\begin{bmatrix}6\\35\end{bmatrix} - 2\begin{bmatrix}6\\44\end{bmatrix} &= 0,\\
20 + 4\begin{bmatrix}4\\33\end{bmatrix} - 10\begin{bmatrix}6\\35\end{bmatrix} + 6\begin{bmatrix}7\\36\end{bmatrix} - 3\begin{bmatrix}6\\44\end{bmatrix} - 4\begin{bmatrix}7\\45\end{bmatrix} &= 0,\\
\begin{bmatrix}4\\33\end{bmatrix} - 3\begin{bmatrix}6\\35\end{bmatrix} &= 0,\\
\begin{bmatrix}5\\34\end{bmatrix} - 2\begin{bmatrix}6\\44\end{bmatrix} &= 0.
\end{aligned}
$$

Proposition. The non-zero $\begin{bmatrix}k\\ij\end{bmatrix}$ of $M = G/K = \mathrm{E}_8 / \mathrm{U}(1) \times \mathrm{SU}(4) \times \mathrm{SU}(5)$ with respect the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, are given explicitly by

$$
\begin{bmatrix}4\\33\end{bmatrix} = 12, \quad \begin{bmatrix}5\\34\end{bmatrix} = 8, \quad \begin{bmatrix}6\\35\end{bmatrix} = 4, \quad \begin{bmatrix}7\\36\end{bmatrix} = 4/3, \quad \begin{bmatrix}6\\44\end{bmatrix} = 4, \quad \begin{bmatrix}7\\45\end{bmatrix} = 2.
$$

• An E_8-invariant metric $(\ ,\ ) = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5_+$ on G/K is **Einstein**, **iff** it is a **positive real solution** of the system:

$$\{E_1 = \bar{r}_1 - \bar{r}_2 = 0,\ E_2 = \bar{r}_2 - \bar{r}_3 = 0,\ E_3 = \bar{r}_3 - \bar{r}_4 = 0,\ E_4 = \bar{r}_4 - \bar{r}_5 = 0\}$$

$$\begin{aligned}
E_1 \quad = \quad & -15{x_2}^3 x_3 x_4 x_5 - 14{x_2}^3 x_4 x_5 - 2{x_2}^3 x_4 - 3{x_2}^2{x_3}^2 x_5 - {x_2}^2 x_3 {x_4}^2 + 60{x_2}^2 x_3 x_4 x_5 \\
& + {x_2}^2 x_3 - 3{x_2}^2{x_4}^2 x_5 + 3{x_2}^2 x_5 + 2x_2{x_3}^2 x_4 x_5 + 2x_2{x_3}^2 x_4 - x_2{x_5}^2(x_2 x_3 - 2x_4) \\
& - 48x_2 x_3 x_4 x_5 + 14x_2 x_4 x_5 + 4x_3{x_4}^2 x_5 = 0,
\end{aligned}$$

$$\begin{aligned}
E_2 \quad = \quad & 6{x_2}^3 x_3 x_4 x_5 + 20{x_2}^3 x_4 x_5 + 5{x_2}^3 x_4 - 6{x_2}^2{x_3}^2 x_5 + 6{x_2}^2{x_4}^2 x_5 - 60{x_2}^2 x_4 x_5 \\
& + 6{x_2}^2 x_5 - 20x_2{x_3}^2 x_4 x_5 - 5x_2{x_3}^2 x_4 + 48x_2 x_3 x_4 x_5 + x_2 x_4{x_5}^2 + 4x_2 x_4 x_5 \\
& - 4x_3{x_4}^2 x_5 = 0,
\end{aligned}$$

$$\begin{aligned}
E_3 \quad = \quad & -12{x_2}^3 x_4 x_5 - 3{x_2}^3 x_4 + 18{x_2}^2{x_3}^2 x_5 - 4{x_2}^2 x_3{x_4}^2 - 48{x_2}^2 x_3 x_5 + 4{x_2}^2 x_3 \\
& - 18{x_2}^2{x_4}^2 x_5 + 60{x_2}^2 x_4 x_5 + 6{x_2}^2 x_5 + 12x_2{x_3}^2 x_4 x_5 + 3x_2{x_3}^2 x_4 \\
& + x_2{x_5}^2(4x_2 x_3 - 3x_4) - 12x_2 x_4 x_5 - 6x_3{x_4}^2 x_5 = 0,
\end{aligned}$$

$$\begin{aligned}
E_4 \quad = \quad & 15{x_2}^3 x_4 - 12{x_2}^2{x_3}^2 x_5 + 14{x_2}^2 x_3{x_4}^2 - 60{x_2}^2 x_3 x_4 + 48{x_2}^2 x_3 x_5 + 6{x_2}^2 x_3 \\
& + 12{x_2}^2{x_4}^2 x_5 - 12{x_2}^2 x_5 + 15x_2{x_3}^2 x_4 - x_2{x_5}^2(14x_2 x_3 + 15x_4) + 6x_3{x_4}^2 x_5 = 0
\end{aligned}$$

A note on the method arising from the twistor fibration

- For any flag manifold $M = G/K$ there exists a natural fibration $\pi : G/K \to G/H$, over an isotropy irreducible symmetric space G/H of compact type. (Burstall-Rawnsley, 90)

- Let $M = G/K = \mathrm{E}_8 / \mathrm{SU}(5) \times \mathrm{SU}(4) \times \mathrm{U}(1)$ with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_5$.
- We do not need the decomposition of $\mathfrak{k}$, thus we set

$$\begin{bmatrix} 2 \\ 11 \end{bmatrix} = \begin{bmatrix} 4 \\ 33 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \end{bmatrix} = \begin{bmatrix} 5 \\ 34 \end{bmatrix}, \begin{bmatrix} 4 \\ 13 \end{bmatrix} = \begin{bmatrix} 6 \\ 35 \end{bmatrix}, \begin{bmatrix} 4 \\ 22 \end{bmatrix} = \begin{bmatrix} 6 \\ 44 \end{bmatrix}, \begin{bmatrix} 5 \\ 14 \end{bmatrix} = \begin{bmatrix} 7 \\ 36 \end{bmatrix}, \begin{bmatrix} 5 \\ 23 \end{bmatrix} = \begin{bmatrix} 7 \\ 45 \end{bmatrix}.$$

- Set

$$\begin{aligned} \mathfrak{h} &= \mathfrak{k} \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_4, \\ \mathfrak{p} &= \mathfrak{m}_1 \oplus \mathfrak{m}_3 \oplus \mathfrak{m}_5. \end{aligned}$$

- Then

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}.$$

• Thus, the coset G/H is a (locally) symmetric space $\Rightarrow$ Since $G = \mathrm{E}_8$ is simply connected, G/H is a symmetric space, namely $G/H = \mathrm{E}_8 / \mathrm{SO}(16)$.

$$\pi : \mathrm{E}_8 / \mathrm{SU}(5) \times \mathrm{SU}(4) \times \mathrm{U}(1) \to \mathrm{E}_8 / \mathrm{SO}(16),$$

with fiber $H/K = \mathrm{SO}(16)/\,\mathrm{SU}(5) \times \mathrm{SU}(4) \times \mathrm{U}(1)$.

• The effective fiber $H'/K' = H/K$ is given by $\mathrm{SO}(16)/\,\mathrm{U}(5) \times \mathrm{SO}(6)$. This is a flag manifold with **two isotropy summands**, i.e.

$$T_o H'/K' = \mathfrak{f} = \mathfrak{f}_1 \oplus \mathfrak{f}_2 = \mathfrak{m}_2 \oplus \mathfrak{m}_4$$

• Therefore H'/K' has only one non-zero structure constant, namely $(c_{22}^4)' = \begin{bmatrix} 4 \\ 22 \end{bmatrix}'$, given by

$$\begin{bmatrix} 4 \\ 22 \end{bmatrix}' = \frac{d_2 d_4}{d_2 + 4d_4} = 60/7.$$

- Since $H' \subset G$ is simple $\Rightarrow$ $\exists\, k > 0$ such that $B_{H'} = k \cdot B_G = k \cdot B$. $\Rightarrow$ $k = 28/60$.
- Then, one can prove that the non-zero triple c^4_{22} of G/K is given by

$$c^4_{22} = \begin{bmatrix} 4 \\ 22 \end{bmatrix} = k \cdot \begin{bmatrix} 4 \\ 22 \end{bmatrix}' \Rightarrow \begin{bmatrix} 4 \\ 22 \end{bmatrix} = \frac{28}{60} \cdot \frac{60}{7} = 4.$$

- By using the Käbler-Einstein metric we express $c^2_{11}, c^4_{13}, c^3_{12}$ and c^5_{14} in terms of c^4_{22} and c^5_{23}:

$$\begin{aligned} c^2_{11} &= 4 + c^4_{22} + 2c^5_{23}, \\ c^3_{12} &= (20 - c^4_{22})/2, \\ c^4_{13} &= (16 - c^4_{22} - 2c^5_{23})/2, \\ c^5_{14} &= 10/3 - c^5_{23}. \end{aligned}$$

- We verify also that $c^3_{12} = \begin{bmatrix} 3 \\ 12 \end{bmatrix} = 8$ $\cdots$ but $c^5_{23} =?$
- This fact shows how **powerful** is the method of Riemannian submersions.

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