Bulk deformation of coisotropic submanifolds

Hông Vân Lê
Institute of Mathematics of ASCR
Zitna 25, 11567 Praha 1, Czech Republic
(joint work with Yong-Geun Oh)

AMeGA and ECI Workshop 2012,
October 20, 2012

• Y.-G. Oh and J.-S. Park, Deformations of coisotropic submanifolds and strong homotopy Lie algebroids, *Invent. Math., 161 (2005) 287- 360*.

On the mathematics side, coisotropic submanifolds are generalization of both symplectic submanifolds and Lagragian submanifolds, on the physics side coisotropic submanifolds serve as boundary condition of the bran theory in physics, according to Park. Oh and Park considered only deformations of coisotropic submanifolds in a symplectic manifold with a fixed symplctic structure. They proposed to consider bulk deformations of coisotropic submanifolds in symplectic manifolds where the symplectic structure is allowed to vary. The first papers discussing such bulk-deformations have been posted in arXiv only this summer. One is written by Schätz and Zambon (arXiv:1207.1696), the second one has been done by Oh and myself (arXiv:1208.3590). In fact, we consider not only bulk deformation of coisotropic submanifolds in a symplectic manifold but in a larger class of l.c.s. manifolds, where the l.c.s. structure is allowed to vary.

1. Locally conformal symplectic manifolds and locally conformal pre-symplectic manifolds

An l.c.s. manifold is a triple $(X^{2n}, \omega, \mathfrak{b})$ where $\mathfrak{b} \in \Omega^1(X^{2n})$, $\omega \in \Omega^2(X^{2n})$ s.t.

$$d\mathfrak{b} = 0,$$

$$\omega^n \neq 0 \text{ and } d\omega + \mathfrak{b} \wedge \omega = 0.$$

L.c.s. manifolds are natural phase spaces of Hamiltonian dynamics, mapping torus of a contactomorphism. They contain the subclass of L.C. K. manifolds.

A triple (Y, ω, b) is called an l.c.p-s. manifold if $\mathfrak{b} \in \Omega^1(X^{2n})$, $\omega \in \Omega^2(X^{2n})$ s.t.

$$d\mathfrak{b} = 0,\ \omega^{k+1} = 0$$

$$\omega^k \neq 0 \text{ and } d\omega + \mathfrak{b} \wedge \omega = 0.$$

Let (Y, ω, b) and (Y', ω', b') be two l.c.p-s. manifolds. A diffeomorphism $\phi : Y \to Y'$

is called l.c.p-s. if there exists $a \in C^\infty(Y, \mathbf{R} \setminus \{0\})$ such that

$$\phi^* \omega' = (1/a)\omega, \quad \phi^* b' = b + d(\ln |a|).$$

2. Characterization of coisotropic submanifolds

A submanifold $Y^{n+k} \subset (M^{2n}, \omega, \mathfrak{b})$ is called coisotropic, if $\forall x \in Y^{n+k}$

$$(T_x Y^{n+k})^\omega \subset T_x Y^{n+k}.$$

$\Leftrightarrow T_x Y^{n+k}$ is maximally degenerate
$\Leftrightarrow rk\, \omega|_{Y^{n+k}} = 2k \Leftrightarrow \omega^{k+1} = 0$.

Example

$$Y^{n+k} = \mathbf{C}^k \oplus \mathbf{R}^{n-k} \subset \mathbf{C}^k \oplus \mathbf{C}^{n-k} = M^{2n}.$$

- $(Y^{n+k}, \omega|_{Y^{n+k}}, \mathfrak{b}|_{Y^{n+k}})$ is an l.c. p-s. manifold.

3. Normal form theorem of a coisotropic submanifold $Y \subset (M^{2n}, \omega, \mathfrak{b})$

$$TY = G \oplus E, \quad E = (TY)^{\omega}.$$

$p_G : TY \to E$ - the associated projection.
$\pi : E^* \to Y$.

$$TE^* \xrightarrow{T\pi} TY \xrightarrow{p_G} E.$$

Let $\widehat{\alpha} \in E^*$ and $\xi \in T_{\widehat{\alpha}}E^*$. We define the one-form θ_G on E^* by

$$\theta_G(\widehat{\alpha}, \xi) := \widehat{\alpha}(p_G \circ T\pi(\xi))$$

Then $\omega_G := \pi^*\omega - d\theta_G - \pi^*b \wedge \theta_G$ is non-degenerate in a neighborhood $U \subset E^*$ of the zero section of the bundle $E^* \to Y$.

Proposition The pair $(U, \omega_G, \mathfrak{b}_U)$ with $\mathfrak{b}_U := \pi^* b|_U$ is an l.c.s. manifold.

Theorem Assume that Y is compact coisotropic submanifold in an l.c.s. $(X, \omega_X, \mathfrak{b})$. There exist an open neighborhood $U \subset X$ of Y, a neighborhood $V \subset E^*$ of the zero section Y, and a l.c.s. diffeomorphism

$$\phi : (U, \omega_X, \mathfrak{b}) \to (V, (\omega_G)|_V, \pi^* b)$$

such that $\phi|_Y = Id$ and

$$\phi^*(\omega_G) = e^{-f} \omega_X$$

for some $f \in C^\infty(U)$.

Example

$Y := (M^{2k}, \omega) \times L^n$, $\omega_Y := e^f \omega$, $b := e^f df$.

$X := M^{2k} \times T^* L^n$,

θ_L -the Louiville 1-form on $T^* L^n$.

$\omega_X := \pi^*(\omega_Y) - d^{\pi^* b}(\theta_L)$.

Then Y is coisotropic in $(X, \omega_X, \mathfrak{b} := \pi^* b)$ and (Y, ω_Y, b) is a l.c.p-s. manifold.

4. (Deformed)-Strongly homotopy Lie algebroid associated with a coisotropic submanifold

Let (Y^{n+k}, ω, b) be a l.c.p-s. manifold. Since $d\omega = -\omega \wedge b$, the distribution $\mathcal{F} := \ker \omega$ is integrable. Hence $(Y^{n+k}, \mathcal{F})$ is a foliated manifold. Denote by $d_{\mathcal{F}}$ the leaf-wise differential.

• The triple $(\mathcal{F}, i : \mathcal{F} \to TY, [,])$ is a Lie algebroid.

• The restriction $\bar{b}$ of b to $\mathcal{F}$ is a closed 1-form in the complex $(\Omega^\bullet(\mathcal{F}), d_{\mathcal{F}})$. Hence $d^{\bar{b}}_{\mathcal{F}} := d_{\mathcal{F}} + \bar{b}\wedge$ satisfies $(d^{\bar{b}}_{\mathcal{F}})^2 = 0$.

• The $\bar{b}$-deformed differential $d_{\mathcal{F}}^{\bar{b}}$ is related to the infinitesimal deformation space of coisotropic submanifolds in a l.c.s. manifold. For this, we introduce the space

$$\mathcal{C}oiso_k = \mathcal{C}oiso_k(X, \omega_X)$$

the set of coisotropic submanifolds with nullity $n-k$ for $0 \leq k \leq n$ and characterize its infinitesimal deformation space at $Y \subset E^*$, the zero section of E^*. By the normal form theorem, the infinitesimal deformation space $T_Y\mathcal{C}oiso_k(X, \omega_X)$ depends only on $(Y, i^*\omega_X)$. An element in $T_Y\mathcal{C}oiso_k(U, \omega_U)$ is a section of the bundle $E^* = T^*\mathcal{F} \to Y$.

• Let $(E \to Y, \rho, [.,.])$ be a Lie algebroid. A b-deformed L_∞-structure over the Lie algebroid is a structure of strong homotopy Lie algebra $(\Gamma(\Lambda^\bullet(E^*))[\underline{1}], \mathfrak{m}_k)$ such that $\mathfrak{m}_1$ is the E-differential ${}^E d^b$.
$(E \to Y, \mathfrak{m}_k)$ is called a b-deformed strong homotopy Lie algebroid.

Theorem Let (Y, ω, b) be a l.c.p.s. manifold and $\Pi : TY = G \oplus T\mathcal{F}$ be a splitting. Then Π canonically induces a structure of strong homotopy Lie algebroid on

$$\left(\bigoplus_\bullet \Omega[1]^\bullet(\mathcal{F}), \{\mathfrak{m}_\ell^b\}_{1 \le \ell < \infty}\right)$$

Furthermore, the isomorphism class of this (b-deformed) strong homotopy Lie algebroid does not depend on the choice of a splitting Π.

Here $\Omega[1]^\bullet(\mathcal{F})$ is the shifted complex of $\Omega^\bullet(\mathcal{F})$, i.e., $\Omega[1]^k(\mathcal{F}) = \Omega^{k+1}(\mathcal{F})$. For $\xi_i \in \Omega^l(\mathcal{F})$ set

$$\mathfrak{m}_1^b(\xi) := (-1)^{|\xi|} d_{\mathcal{F}}^b(\xi)$$

$$\mathfrak{m}_2^b(\xi_1, \xi_2) := (-1)^{|\xi_1|(|x_2|+1)} (P_\omega \rfloor d_G^b \xi_1 \wedge d_G^b \xi_2),$$

where $P_\omega \in \Lambda^2 G$ is dual to the restriction of ω to G, in particular $\omega(P_\omega) = k$.

On the un-shifted group $\Omega^\bullet(\mathcal{F})$, $d^b_{\mathcal{F}}$ defines a differential of degree 1 and $\{\cdot,\cdot\}_\omega$ is a graded bracket of degree 0 and $\mathfrak{m}^b_\ell$ is a map of degree $2-\ell$.

5. Moduli problem and the Kuranishi map

Master equation The graph Γ_s of a section $s : Y \to (U \subset E^*, \omega_G, \pi^* b)$ is coisotropic iff

$$\omega_G(s) := \omega|_Y - d^b(p_G^*(s)) \in \Omega^2(Y)$$

satisfies

$$(\omega_G(s))^{k+1} = 0.(**)$$

The equation of the formal power series solutions $\Gamma \in \Omega^1(\mathcal{F})$ of $(**)$ is given by

$$\sum_{\ell=1}^{\infty} \frac{1}{\ell!} \mathfrak{m}_\ell^b(\Gamma, \cdots, \Gamma) = 0 \in \Omega^2(\mathcal{F}),$$

$$\Gamma := \sum_{k=1}^{\infty} \varepsilon^k \Gamma_k.$$

Kuranishi map Solving the equation for formal solution of the deformation of coisotropic submanifolds we discover new non-linear condition posed on an infinitesimal deformation $\Gamma \in \Omega^1(\mathcal{F})$ of a coisotropic submanifold Y.

Solving the equation for the summand associated to ε we obtain:

$$d^b_{\mathcal{F}}\Gamma_1 = 0.(*)$$

• (*)is the linearized equation of the master equation, i.e. the equation for $\Gamma_1 \in \Omega^1(\mathcal{F})$ to be a formal infinitesimal coisotropic deformation of $Y \subset (X, \omega_X, \mathfrak{b})$.

• Two solutions of the equation (*) are infinitesimally Hamiltonian equivalent if and only if they are cohomologous as elements in $\Omega_b^1(Y,\omega)$. Consequently, the set of equivalence classes of the infinitesimally Hamiltonian equivalent solutions of the linearized equations is $H_b^1(Y,\omega)$.

• The set of the infinitesimal l.c.s. equivalence classes of the solutions ξ of the equation (*) has 1-1 correspondence with $H_b^1(Y,\omega)/i^*(H_b^1(Y))$ if $[\omega] \neq 0$ in $H_b^2(Y)$ and $H_b^1(Y,\omega)/(i^*(H_b^1(Y)) + \langle\theta|_{\mathcal{F}}\rangle_{\otimes\mathbf{R}})$ if $\omega = d^b\theta$.

• Solving the equation for the summand associated to ε^2 we obtain

$$-d^{\bar{b}}_{\mathcal{F}}\Gamma_2 = \frac{1}{2}P_\omega\rfloor(d^{\bar{b}}_{\mathcal{F}}\Gamma_1)^2$$

Since $\mathfrak{m}_1^b$ is a derivation of $\mathfrak{m}_2^b$ the map

$$\Omega^1(\mathcal{F}) \times \Omega^1(\mathcal{F}) \to \Omega^2(\mathcal{F}),$$

$$(\Gamma_1, \Gamma_2) \mapsto \frac{1}{2}P_\omega\rfloor(d^b\Gamma_1 \wedge d^b\Gamma_2)$$

induces the Kuranishi-Gerstenhaber bracket

$$KG : H_b^1(Y,\omega) \times H_b^1(Y,\omega) \to H_b^2(Y,\omega),$$

$$([\Gamma_1],[\Gamma_2]) \mapsto \frac{1}{2}[\mathfrak{m}_2^b(\Gamma_1,\Gamma_2)].$$

Since $\mathfrak{m}_2^b$ is symmetric, the KG bracket is defined by the Kuranishi map

$$Kr : H_b^1(Y,\omega) \to H_b^2(Y,\omega),$$

$$[\Gamma_1] \mapsto [\mathfrak{m}_2^b(\Gamma_1,\Gamma_1)].$$

• The moduli problem is formally unobstructed only if Kr vanishes. In particular, the formal moduli problem is unobstructed if $H_b^2(Y,\omega) = \{0\}$.

6. Deformations of l.c.s. forms

We call a smooth one-parameter family $(X, \omega_t, \mathfrak{b}_t)$ of l.c.s structures a bulk-deformation.

• Let $(X, \omega_t, \mathfrak{b}_t)$ be a bulk-deformation of l.c.s. structure on X with $(\omega_0, \mathfrak{b}_0) = (\omega_X, \mathfrak{b})$. Denote

$$\left.\frac{\partial \omega_t}{\partial t}\right|_{t=0} = \kappa, \quad \left.\frac{\partial \mathfrak{b}_t}{\partial t}\right|_{t=0} = \mathfrak{c}$$

Then $(\kappa, \mathfrak{c})$ satisfies

$$d^{\mathfrak{b}}\kappa = -\mathfrak{c} \wedge \omega_X, \quad d\mathfrak{c} = 0.$$

• We denote by $\mathrm{Def}(X, \omega_X, \mathfrak{b})$ the set of equivalence classes of infinitesimal deformations of $(X, \omega_X, \mathfrak{b})$. Set

$$L : H^1(X, \mathbf{R}) \to H^3_{\mathfrak{b}}(X, \mathbf{R}),\ [\alpha] \mapsto [\alpha] \wedge [\omega_X].$$

$$\mathrm{Def}(X, \omega_X, \mathfrak{b}) = \ker L \oplus H^2_{\mathfrak{b}}(X)/\langle \omega_X \rangle_{\otimes \mathbf{R}}$$

7. Bulk deformations of coisotropic submanifold and Zambon's example

Assume that Y is a coisotropic submanifold of $(U, \omega_U, d^{\pi^* b}\theta_G)$. A deformation $\Gamma_t : Y \to U$ is called a bulk coisotropic deformation, if there exists a family of l.c.p-s. form $(\bar{\omega}_t, b_t)$

of constant rank on Y with $\bar{\omega}_0 = i^*\omega_U$, $b_0 = b$ and for each $t \in [-\varepsilon, \varepsilon]$ (the graph of) Γ_t is coisotropic in $(U, \pi^*\bar{\omega}_t, d^{\pi^* b_t}\theta_G)$.

Zambon's example (DGA, 2008, arxiv:1207.1696) Let (Y, ω) be the standard 4-torus $T^4 = \mathbf{R}^4/\mathbf{Z}^4$ with coordinates (y^1, y^2, q^1, q^2) with the pre-symplectic form

$$\omega_Y = \bar{\omega}_0 = dy^1 \wedge dy^2,\ b_0 = 0.$$

The null foliation $\mathcal{F}$ is provided by the 2-tori

$$\{y^1 = const,\ y^2 = const\}.$$

It also carries the transverse foliation given by

$$\{q^1 = const, \, q^2 = const\}.$$

The canonical symplectic thickening is given by

$$\begin{aligned} E^* &= T^4 \times \mathbf{R}^2 = T^2 \times T^*(T^2), \\ \omega &= dy^1 \wedge dy^2 + (dq^1 \wedge dp^1 + dq^2 \wedge dp^2), \end{aligned}$$

where p^1, p^2 are the canonical conjugate coordinates of q^1, q^2.

Oh-Park (Inventiones, 2005) proved that the one-form

$$\Gamma_1 = \sin(2\pi y^1)dq^1 + \sin(2\pi y^2)dq^2,$$

is obstructed by showing that $Kr([\Gamma_1]) \neq 0$.

Theorem Γ_1 is formally bulk obstructed. (H. V. Lê and Y-G. Oh, arxiv:1208.3590, which is stronger than the similar statement due to Schätz and Zambon, arxiv:1207.1696.)

8. Open questions and progress

- Understanding the smooth moduli space. First to understand the Zariski tangent space $H_b^1(\mathcal{F})$. (We are lead to discover a new smooth Hodge leafwise decomposition for compact foliated Riemannian manifolds).

- Understanding the strongly homotopy Lie algebroid associated to an l.c.p-s. manifolds using representation theory ans spectral sequences.

- Extend the above ideas to Poisson manifolds (Cattaneo-Felder, Zambon-Schätz, etc.)

- For more open problems, see Oh-Park, Inventiones, 2005 and Le-Oh, arxiv:1207.1696.

Thank you!