Are weak equivalences finitely accessible?

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Proposition 1. The category $\mathcal{W}$ of weak equivalences in simplicial sets is $\aleph_1$-accessible.

Proof. $\mathcal{W}$ is given by a pullback

$$\begin{array}{ccc} \mathcal{S}^{\rightarrow} & \xrightarrow{F} & \mathcal{S}^{\rightarrow} \\ \uparrow & & \uparrow \\ \mathcal{W} & \longrightarrow & \mathcal{F}_0 \end{array}$$

where $\mathcal{F}_0$ is the category of trivial fibrations and F sends a morphism to the fibration in its (trivial cofibration, fibration) factorization. Since $\mathcal{F}_0$ is $\aleph_1$-accessible and F preserves filtered colimits and $\aleph_1$-presentable objects, $\mathcal{W}$ is $\aleph_1$-accessible.

The same is true for any finitely combinatorial model category. For simplicial sets, there is a better result.

Proposition 2. The category $\mathcal{W}$ of weak equivalences in simplicial sets is ∞, ω-elementary.

Proof. $\mathcal{W}$ is given by a pullback

$$\begin{array}{ccc} \mathcal{S}^{\to} & \xrightarrow{\pi_* R} & (Gr^{\omega})^{\to} \\ \uparrow & & \uparrow \\ \mathcal{W} & \longrightarrow & \mathcal{I}so \end{array}$$

where R is a fibrant replacement functor and π_* the homotopy group functor. Thus $\mathcal{W}$ is a limit of finitely accessible categories (and finitely accessible functors).

Following Makkai and Paré, $\mathcal{W}$ can be axiomatized by universaly quantified implications of $\exists \bigvee \wedge$ formulas. This was observed by Beke.

The same is true for homology isomorphisms of chain complexes of modules.

Problem. Is $\mathcal{W}$ finitely accessible?

Proposition 3. Let $\mathcal{K}$ be a finitely combinatorial model category. Then the category of trivial cofibrations between cofibrant objects is finitely preaccessible.

Proof follows from the fat small object argument applied to $\mathcal{K}^{\to}$ with generating arrows $(i, i) : \mathrm{id} \to \mathrm{id}$ for each generating cofibration in $\mathcal{K}$ and $(\mathrm{id}, j) : \mathrm{id} \to j$ for each generating trivial cofibration.

Corollary 1. The category of trivial cofibrations in simplicial sets is finitely accessible.

This was proved by Joyal and Wraith in 1984.

Corollary 2. Let $\mathcal{K}$ be a finitely combinatorial category. Then the category of cofibrations between cofibrant objects is finitely accessible.

For a finitely cogenerated cotorsion theory $(\mathcal{A}, \mathcal{B})$, we get the category of $\mathcal{A}$-objects with $\mathcal{A}$-monomorphisms. This category was considered by Baldwin, Eklof and Trlifaj. They studied when it is an abstract elementary class.

I do not know any abstract elementary class which is not ∞, ω-elementary.

Proposition 4. Let $\mathcal{K}$ be a finitely combinatorial model category where 1 is finitely presentable and any morphism $\mathcal{K} \to 1$ splits by a cofibration. Then the category of acyclic objects is finitely accessible.

In particular, acyclic objects in simplicial sets are finitely accesible. This was proved by Joyal and Wraith.

Proposition 5. Any trivial fibration in simplicial sets is a filtered colimit of finitely presentable weak equivalences.

Proof. If $f : A \to B$ is a trivial fibration with B finitely presentable, the result follows from Proposition 4. applied to $\mathcal{S} \downarrow B$. The general case is obtained by using filtered colimits and pullbacks:

$$\begin{array}{ccc} \mathcal{A} \to & \xrightarrow{\quad f \quad} & \mathcal{B} \to \\ \uparrow & & \uparrow \\ \mathcal{A}_i & \longrightarrow & \mathcal{B}_i \end{array}$$

Proposition 6. Equivalences of categories are finitely accessible.

Proof. It follows from the fat small object argument applied to $\mathrm{Cat}^{\to}$ with generating arrows $(i, i) : \mathrm{id}_1 \to \mathrm{id}_E$, $(\mathrm{id}, i) : \mathrm{id}_1 \to i$ and $(i, \mathrm{id}) : \mathrm{id}_1 \to k$ where $i : 1 \to E$ sends 1 to the free living equivalence E and k splits i.