k-Dirac operator and the boundary value problem

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- ε_j. denotes the multiplication by the Clifford number ε_j.

k-Dirac operator (in the parabolic setting).

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- Levi subgroup of P is isomorphic to $\mathrm{GL}(k, \mathbb{R}) \times \mathrm{SO}(n)$ with isomorphisms $\mathfrak{g}_{-1} \cong (\mathbb{R}^k)^* \otimes \mathbb{R}^n, \mathfrak{g}_{-2} \cong \Lambda^2(\mathbb{R}^k)^*$.

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- k-Dirac operator is the invariant operator

$$D : \Gamma(\mathcal{S}_\pm) \to \Gamma(\mathbb{R}^k \otimes \mathcal{S}_\pm)$$

where $\mathcal{S}_\pm$ is the homogeneous vector bundle associated $\mathbb{S}_\pm$ (n is even) over G/P.

Local expression for D.

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$$D(\varphi) = (D_1\varphi, \dots, D_k\varphi)$$
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- Question: Is there also an arrow in the other direction?
- Answer: Yes, there is.

The set of initial conditions for the k-Dirac operator (in the Euclidean setting).

- Let $f \in \mathcal{C}^\infty(M(n,k,\mathbb{R}),\mathbb{S}_\pm)$ be a solution of $\partial f = 0$. Then f is called a *monogenic spinor*.

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- f is of the form $g + x_{1i}f_i + x_{1i}x_{1j}f_{ij} + \ldots$ where each $f_{i_1 \ldots i_r}$ is a homogeneous spinor of degree $m - r$ which depends only $x_{\alpha i}, \alpha \geq 2$.

Parabolic operator from the Euclidean k-Dirac op.

Theorem

The monogenic spinor f exists iff $\forall i, j : [\tilde{\partial}_i, \tilde{\partial}_j]g = 0$ *where* $\tilde{\partial}_i = \sum_{\alpha \geq 2} \varepsilon_\alpha . \partial_{\alpha i}$.

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Monogenic spinors in the parabolic setting.

Theorem

Let $\psi \in \Gamma(\mathcal{S}_\pm)$ be an arbitrary real analytic spinor given on an open subset U' of the set U, i.e. ψ depends only on $x_{\alpha i}, \alpha \geq 2$. Then ψ extends to a unique monogenic spinor Ψ on a small open neighbourhood of U', i.e. $D\Psi = 0$ and $\Psi|_{U'} = \psi|_{U'}$.

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- $\psi \in \Gamma(\mathcal{S}_\pm) : \psi = \psi_0 + \psi_1 + \dots$ where ψ_i is homogeneous of degree i.
- D is homogeneous operator of the degree (-1). Thus $D\psi = 0$ iff $D\psi_m = 0$ for each homogeneous piece ψ_m of ψ.

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- This implies also that

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where $\tilde{D}_i = \sum_{\alpha\geq 2} \varepsilon_\alpha L_{\alpha i}$.

Monogenic spinors in the parabolic setting.

- Let $\psi \in \Gamma(\mathcal{S}_\pm)$ be a monogenic spinor ($\forall i : D_i\psi = 0$).
- Then $\{D_i, D_j\}\psi = 0$ and thus also

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- Get PDE on $V := \{x_{11} = \ldots = x_{1k} = 0\}$.

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where $f_i^j \in \mathbb{C}[y_{rs}]$, $deg(f_j^i) = j$ and each spinor $\varphi_j^i \equiv \varphi_j^i(x_{\alpha i}), \alpha \geq 2, deg_w(\varphi_j^i) = m - 2j$.

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- If φ is a solution of $(\triangle)$, then for each $i : \varphi_m^i$ satisfies $[\tilde{\partial}_i, \tilde{\partial}_j]\varphi_m^i = 0$.
- Thus for each $i : \varphi_m^i$ extends to a unique solution of the k-Dirac operator in the Euclidean setting.

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Lemma

Let $\phi \in \mathcal{C}^\infty(M(n,k,\mathbb{R}), \mathbb{S}_\pm)$ be a (homogeneous) solution of $\partial\phi = 0$ and $f \in \mathbb{C}[y_{rs}]$ be an arbitrary homogeneous polynomial. Then there exists a monogenic spinor $\Phi = f\phi + l.o.t.$ in the parabolic setting ($D\Phi = 0$) where $l.o.t.$ stands for a spinor whose components are polynomials of homogeneity strictly smaller then $deg(f)$ in y_{rs}-variables. (Any monogenic spinor is of this form).

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- Call $f\phi$ the *leading term* of Φ.
- The end of the proof of the theorem:
- The uniqueness: Let ψ be a monogenic spinor such that $\psi|_U = 0$. Then $\psi|_V$ is a solution of $(\triangle)$ so by Cartan-Kähler theorem $\psi = 0$ on V. By the previous lemma ψ is determined by its leading term which is zero. Thus $\psi = 0$ and so any monogenic spinor is uniquely determined by its restriction to U.

The end of the proof of the theorem.

- The existence of an extension: Let ψ be a homogeneous spinor ($D\psi = 0$) in $x_{\alpha i}$-variables, $\alpha \geq 2$. Let $\bar{\psi}$ the unique solution of ($\triangle$) determined by ψ given by the Cartan-Kähler theorem.

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- Let us write $\bar{\psi} = f^i_m \bar{\psi}^i_m + l.o.t.$ with $deg(f^i_m) = m$. Call $f^i_m \bar{\psi}^i_m$ the leading term of $\bar{\psi}$.

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- Let Ψ be a monogenic spinor given by the leading term of $\bar{\psi}$ from the previous lemma.

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- Let Ψ be a monogenic spinor given by the leading term of $\bar{\psi}$ from the previous lemma.
- Then the leading terms of $\bar{\psi}$ and $\Psi|_V$ agree. Might happen that $\bar{\psi} - \Psi|_V \neq 0$. Nevertheless, the difference $\bar{\psi} - \Psi|_V$ is again a solution of ($\triangle$) and so you can apply the lemma again to the leading term of $\bar{\psi} - \Psi|_V$. By induction on m the existence of an extension follows. □

Thank you for your attention!