Conformally flat Lorentzian manifolds with special holonomy groups

Anton S. Galaev

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Abstract

The local classification of conformally flat Lorentzian manifolds with special holonomy groups is obtained. The corresponding local metrics are certain extensions of Riemannian spaces of constant sectional curvature to Walker metrics.

Keywords: Lorentzian manifold, special holonomy group, Walker manifold, conformally flat manifold, Nordström's gravity

1 Introduction and the main results

It is known [20] that a conformally flat Riemannian manifold is either a product of two spaces of constant sectional curvature, or it is a product of a space of constant sectional curvature with an interval, or its restricted holonomy group is the identity component of the orthogonal group. The last condition represents the general case and among various manifolds satisfying the last condition one can emphasize only the spaces of constant sectional curvature.

One says that the connected holonomy group of an indecomposable pseudo-Riemannian manifold is special if it is different from the connected component of the pseudo-orthogonal group [6]. In this case the holonomy group defines a special geometry on the manifold. For example, a pseudo-Riemannian manifold of signature (r, s) is pseudo-Kählerian if and only if its holonomy group is contained in $\mathrm{U}(\frac{r}{2}, \frac{s}{2})$. We see that there are no conformally flat Riemannian manifolds with special holonomy groups.

In the case of pseudo-Riemannian manifolds, the holonomy group can be weakly irreducible, this means that it does not preserve any non-degenerate proper vector subspace of the tangent space, and not irreducible in the same time, i.e. it may preserve a degenerate vector subspace of the tangent space.

The main result of the present paper is the complete local description of conformally flat Lorentzian manifolds (M, g) with weakly irreducible not irreducible holonomy groups, which are the only special holonomy groups of Lorentzian manifolds. Let $\dim M = n + 2 \geq 4$. The holonomy algebra, i.e. the Lie algebra of the holonomy group, $\mathfrak{g} \subset \mathfrak{so}(1, n+1)$ of such manifold preserves an isotropic line of the tangent space, which is identified with the Minkowski space $\mathbb{R}^{1,n+1}$. Hence $\mathfrak{g}$ is contained in the maximal subalgebra of $\mathfrak{so}(1, n+1)$ preserving an isotropic line. This algebra is denoted by $\mathfrak{sim}(n)$ and it admits the decomposition

$$\mathfrak{sim}(n) = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n.$$

The classification of the Lorentzian holonomy algebras $\mathfrak{g} \subset \mathfrak{sim}(n)$ can be found in [13]. Any manifold (M, g) with such holonomy algebra (locally) admits a distribution ℓ of isotropic lines.

Such manifolds are called the Walker manifolds [5]. On any such manifold (M,g) there exist local coordinates $v, x^1, ..., x^n, u$ such that the metric g has the form

$$g = 2dvdu + h + 2Adu + H(du)^2, \qquad (1)$$

where $h = h_{ij}(x^1,...,x^n,u)dx^i dx^j$ is an u-dependent family of Riemannian metrics, $A = A_i(x^1,\dots,x^n,u)dx^i$ is an u-dependent family of one-forms, and H is a local function on M. The vector field ∂_v defines the parallel distribution of isotropic lines. An important class of Walker manifolds form pp-waves that are given locally by (1) with $A=0$, $h=\sum_{i=1}^n (dx^i)^2$, and $\partial_v H = 0$, see [21]. The pp-waves are exactly Walker manifolds with the commutative holonomy algebra $\mathfrak{g} \subset \mathbb{R}^n \subset \mathfrak{sim}(n)$.

We use the Einstein summation convention for the indeces $i,j,k=1,...,n$; we use the denotation $\dot f = \partial_u f$ for any function f.

On a Walker manifold (M,g) we define the canonical function λ from the equality

$$\mathrm{Ric}\, p = \lambda p,$$

where p is any local vector field tangent to the distribution ℓ, and Ric is the Ricci operator. If the metric g is written in the form (1), then $\lambda = \frac{1}{2}\partial_v^2 H$, and the scalar curvature of g satisfies

$$s = 2\lambda + s_0,$$

where s_0 is the scalar curvature of h. The form of a conformally flat Walker matric will depend on the vanishing of the function λ. In the general case we obtain the following result.

Theorem 1 *Let (M,g) be a conformally flat (i.e. with zero Weyl curvature tensor) Walker Lorentzian manifold of dimension $n+2 \geq 4$. Then in a neighborhood of each point of M there exist coordinates $v, x^1, ..., x^n, u$ such that*

$$g = 2dvdu + \Psi \sum_{i=1}^n (dx^i)^2 + 2Adu + (\lambda(u)v^2 + vH_1 + H_0)(du)^2,$$

where

$$\Psi = \frac{4}{\left(1-\lambda(u)\sum_{k=1}^n (x^k)^2\right)^2},$$

$$A = A_i dx^i, \quad A_i = \Psi\left(-4C_k(u)x^k x^i + 2C_i(u)\sum_{k=1}^n (x^k)^2\right),$$

$$H_1 = -4C_k(u)x^k\sqrt{\Psi} - \partial_u \ln\Psi + K(u),$$

$$H_0(x^1,...,x^n,u) = \begin{cases} \frac{4}{\lambda^2(u)}\Psi\sum_{k=1}^n C_k^2(u) + \sqrt{\Psi}\left(a(u)\sum_{k=1}^n (x^k)^2 + D_k(u)x^k + D(u)\right), & \text{if } \lambda(u) \neq 0, \\ 16\left(\sum_{k=1}^n (x^k)^2\right)^2 \sum_{k=1}^n C_k^2(u) + \tilde a(u)\sum_{k=1}^n (x^k)^2 + \tilde D_k(u)x^k + \tilde D(u), & \text{if } \lambda(u) = 0 \end{cases}$$

for some functions $\lambda(u)$, $a(u)$, $\tilde a(u)$, $C_i(u)$, $D_i(u)$, $D(u)$, $\tilde D_i(u)$, $\tilde D(u)$.
The scalar curvature of g equals to $-(n-2)(n+1)\lambda(u)$.

If the function λ is locally constantly zero, or it is non-vanishing, then the above metric may be simplified.

Theorem 2 *Let (M,g) be a conformally flat Walker Lorentzian manifold of dimension $n+2 \geq 4$.*

1) If the function λ is non-vanishing at a point, then in a neighborhood of this point there exist coordinates $v, x^1, ..., x^n, u$ such that

$$g = 2dvdu + \Psi \sum_{i=1}^{n} (dx^i)^2 + (\lambda(u)v^2 + vH_1 + H_0)(du)^2,$$

where

$$\Psi = \frac{4}{\left(1 - \lambda(u) \sum_{k=1}^{n} (x^k)^2\right)^2},$$

$$H_1 = -\partial_u \ln \Psi, \quad H_0 = \sqrt{\Psi} \left(a(u) \sum_{k=1}^{n} (x^k)^2 + D_k(u)x^k + D(u) \right).$$

2) If $\lambda \equiv 0$ in a neighborhood of a point, then in a neighborhood of this point there exist coordinates $v, x^1, ..., x^n, u$ such that

$$g = 2dvdu + \sum_{i=1}^{n} (dx^i)^2 + 2Adu + (vH_1 + H_0)(du)^2,$$

where

$$A = A_i dx^i, \quad A_i = C_i(u) \sum_{k=1}^{n} (x^k)^2, \quad H_1 = -2C_k(u)x^k$$

$$H_0 = \sum_{k=1}^{n} (x^k)^2 \left(\frac{1}{4} \sum_{k=1}^{n} (x^k)^2 \sum_{k=1}^{n} C_k^2(u) - (C_k(u)x^k)^2 + \dot{C}_k(u)x^k + a(u) \right) + D_k(u)x^k + D(u).$$

In particular, if all $C_i \equiv 0$, then the metric can be rewritten in the form

$$g = 2dvdu + \sum_{i=1}^{n} (dx^i)^2 + a(u) \sum_{k=1}^{n} (x^k)^2 (du)^2. \qquad (2)$$

Thus Theorem 2 gives the local form of a conformally flat Walker metric in the neighborhoods of any point where λ is non-zero or constantly zero. Such points represent a dense subset of the manifold. Theorem 1 describes also the metric in the neighborhoods of points at that the function λ vanishes, but it is not locally zero, i.g. in the neighborhoods of isolated zero points of λ.

Next, we find the holonomy algebras of the obtained metrics and check when the metrics are decomposable.

Theorem 3 *Let (M, g) be as in Theorem 1.*

1) *The manifold (M, g) is locally indecomposable if and only if there exists a coordinate system with one of the properties:*

- $\dot{\lambda} \not\equiv 0$,
- $\dot{\lambda} \equiv 0$, $\lambda \neq 0$, *i.e. g can be written as in part 1) of Theorem 2, and $\sum_{k=1}^{n} D_k^2 + (a + \lambda D)^2 \not\equiv 0$,*
- $\lambda \equiv 0$, *i.e. g can be written as in part 2) of Theorem 2, and $\sum_{k=1}^{n} C_k^2 + a^2 \neq 0$.*

Otherwise, the metric can be written in the form

$$g = \Psi \sum_{k=1}^{n} (dx^k)^2 + 2dvdu + \lambda v^2 (du)^2, \quad \lambda \in \mathbb{R}.$$

The holonomy algebra of this metric is trivial if and only if $\lambda = 0$. *Otherwise, the holonomy algebra is isomorphic to* $\mathfrak{so}(n) \oplus \mathfrak{so}(1,1)$.

2) *Suppose that* (M, g) *is locally indecomposable. Then its holonomy algebra is isomorphic to* $\mathbb{R}^n \subset \mathfrak{sim}(n)$ *if and only if* $\lambda^2(u) + \sum_{k=1}^{n} C_k^2(u) \equiv 0$ *for all coordinate systems. In this case* (M, g) *is a pp-wave, and* g *is given by* (2). *Otherwise, the holonomy algebra is isomorphic to* $\mathfrak{sim}(n)$.

In Section 2, the result of Kurita [20] is extended to the case of pseudo-Riemannian manifolds. In Section 3, an expression for the curvature tensor and the Weyl conformal curvature tensor W for a Walker metric is given. It seems that these expressions are given here for the first time. Sections 4, 5 and 6 are dedicated to the proofs of the main theorems. We rewrite the equation $W = 0$ as a system of partial differential equations and find appropriate systems of coordinates such that the complete solution of this system can be found. In Section 7, the Ricci operator for the obtained metrics is computed.

In Section 8, the case of dimension 4 is considered. Possible holonomy algebras of conformally flat 4-dimensional Lorentzian manifolds are classified also in [9]. The metric (2) in dimension 4 is given in [25]. In [9], it is posed the problem to construct an example of conformally flat metric with the holonomy algebra $\mathfrak{sim}(2)$ (which is denoted in [9] by R_{14}). An attempt to construct such metric was done in [17]. We show that the metric constructed there is in fact decomposable and its holonomy algebra is $\mathfrak{so}(1,1) \oplus \mathfrak{so}(2)$. Thus in this paper we get conformally flat metrics with the holonomy algebra $\mathfrak{sim}(n)$ for the first time, and even more, we find all such metrics.

The field equations of Nordström's theory of gravitation, which appeared before Einstein's theory, are the following:

$$W = 0, \quad s = 0,$$

see [22, 23, 26]. All metrics from Theorem 1 in dimension 4 and metrics from part 2) of Theorem 2 in bigger dimensions provide examples of solutions of these equations. Thus we have found all solutions to Nordström's gravity with holonomy algebras contained in $\mathfrak{sim}(n)$. Similarly, the Einstein equation on Lorentzian manifolds with such holonomy algebras was studied in [18]. In this case it is impossible to obtain the complete solution, but the examples of solutions have interesting physical interpretations [18].

An important fact is that a simply connected conformally flat spin Lorentzian manifold admits the space of conformal Killing spinors of maximal dimension [3].

It would be interesting to obtain examples of conformally flat Lorentzian manifolds satisfying some global geometric properties, e.g. important are globally hyperbolic Lorentzian manifolds with special holonomy groups [4].

The projective equivalence of 4-dimensional conformally flat Lorentzian metrics with special holonomy algebras was studied recently in [8]. There are many interesting works about conformally flat (pseudo-)Riemannian, and in particular Lorentzian manifolds. Let us mention the works [1, 19, 24, 7, 10, 11, 12].

The results of this paper are used in [2] for the classification of Lorentzian manifolds satisfying the condition $\nabla^2 R = 0$.

2 Decomposability of conformally flat pseudo-Riemannian manifolds

In [20], Kurita proved the following theorem for the case of Riemannian manifolds.

Theorem 4 *Let (M,g) be an n-dimensional conformally flat Riemannian manifold. Then its local restricted holonomy group H_x $(x \in M)$ is in general* $\mathrm{SO}(n)$*. If $H_x \neq \mathrm{SO}(n)$, then for some coordinate neighborhood U of x one of the following holds:*

1) H_x is identity and the metric is flat in U;

2) $H_x = \mathrm{SO}(k)\times\mathrm{SO}(n-k)$ and U is a direct product of a k-dimensional manifold of constant sectional curvature K and an $(n-k)$-dimensional manifold of constant sectional curvature $-K$ $(K \neq 0)$;

3) $H_x = \mathrm{SO}(n-1)$ and U is a direct product of a straight line (or a segment) and an $(n-1)$-dimensional manifold of constant sectional curvature.

We generalize this theorem for the case of pseudo-Riemannian manifolds. We also make it more precise.

Theorem 5 *Let (M,g) be a conformally flat pseudo-Riemannian manifold of signature (r,s) with the restricted holonomy group* $\mathrm{Hol}^0(M,g)$*. If (M,g) is not flat, then one of the following holds:*

1) $\mathrm{Hol}^0(M,g) = \mathrm{SO}(r,s)$*;*

2) $\mathrm{Hol}^0(M,g)$ *is weakly irreducible and not irreducible (in particular, it preserves a degenerate subspace of the tangent space);*

3) $\mathrm{Hol}^0(M,g) = \mathrm{SO}(r_1,s_1)\times\mathrm{SO}(r-r_1,s-s_1)$ *and each point $x \in M$ has a neighborhood that is either flat or it is a product of a pseudo-Riemannian manifold of constant sectional curvature K and signature (r_1,s_1) and a pseudo-Riemannian manifold of constant sectional curvature $-K$ $(K \neq 0)$ and signature $(r-r_1,s-s_1)$;*

4) $\mathrm{Hol}^0(M,g) = \mathrm{SO}(r-1,s)$ *(resp., $H_x = \mathrm{SO}(r,s-1)$) and each point $x \in M$ has a neighborhood that is either flat or it is a product of a pseudo-Riemannian manifold of constant sectional curvature and signature $(r-1,s)$ (resp., $(r,s-1)$) and the space $(L,-(dt)^2)$ (resp., $(L,(dt)^2)$), L is the straight line or a segment.*

Proof. Let (M,g) be a pseudo-Riemannian manifold of signature (r,s) and dimension $d=r+s$. The vector bundle $\mathfrak{so}(TM)$ of skew-symmetric endomorphisms of the tangent bundle TM can be identified with the space of bivectors $\wedge^2 TM$ in such a way that

$$(X\wedge Y)Z = g(X,Z)Y - g(Y,Z)X$$

for all vector fields X,Y,Z on M. The Weyl tensor W of the pseudo-Riemannian manifold (M,g) is defined by the equality

$$W = R + R_L, \tag{3}$$

where the tensor R_L is defined by

$$R_L(X,Y) = LX\wedge Y + X\wedge LY, \tag{4}$$

$$L = \frac{1}{d-2}\left(\mathrm{Ric} - \frac{s}{2(d-1)}\,\mathrm{id}\right)$$

is the Schouten tensor and s is the scalar curvature.

Suppose that the restricted holonomy group $\mathrm{Hol}^0(M,g)$ is not weakly irreducible. The Wu decomposition Theorem [27] states that each point of M has a neighborhood U such that $(U, g|_U)$ is a product

$$(U, g|_U) = (M_1 \times M_2, g_1 + g_2)$$

of two pseudo-Riemannian manifolds (M_1, g_1) and (M_2, g_2). Let d_1 and d_2 be the dimensions of these manifolds. For the curvature tensors, Ricci operators and the scalar curvatures it holds

$$R = R_1 + R_2, \quad \mathrm{Ric} = \mathrm{Ric}_1 + \mathrm{Ric}_2, \quad s = s_1 + s_2.$$

First suppose that $d \geq 4$. In this case $W = 0$ and we get

$$R_1 + R_2 = -R_L. \tag{5}$$

Assume that $d_1 \geq d_2$ and $d_1 \geq 2$. The curvature tensor R_1 can be written in the form $R_1 = W_1 - R_{L_1}$. Considering (5) restricted to TM_1, we get that $W_1 = 0$ and

$$\frac{1}{d_1-2}\left(\mathrm{Ric}_1 - \frac{s_1}{2(d_1-1)}\,\mathrm{id}\right) = \frac{1}{d-2}\left(\mathrm{Ric}_1 - \frac{s_1+s_2}{2(d-1)}\,\mathrm{id}\right). \tag{6}$$

If $d_2 \geq 2$, then taking the trace in (6), we get

$$\frac{s_1}{d_1(d_1-1)} = -\frac{s_2}{d_2(d_2-1)}.$$

Since s_1 is a function on M_1 and s_2 is a function on M_2, the both functions must be constant. Substituting the last equality back to (6), we obtain

$$\mathrm{Ric}_1 = \frac{s_1}{d_1}\,\mathrm{id}\,. \tag{7}$$

Next,

$$R_1(X,Y) = \frac{s_1}{d_1(d_1-1)} X \wedge Y. \tag{8}$$

The same holds for the second manifold. For the sectional curvatures we get

$$k_1 = \frac{s_1}{d_1(d_1-1)} = -\frac{s_2}{d_2(d_2-1)} = -k_2.$$

If $d_2 = 1$, then (6) is equivalent to (7) and this implies (8). From this and the Schur Theorem it follows that k_1 is constant. If $d_1 = 2$, then the curvature tensor R_1 satisfies $R_1(X,Y) = fX \wedge Y$ for some function f on M_1. The proof in this case is the same.

If $d = 3$, then $d_1 = 2$ and $d_2 = 1$. It holds $R = R_1$ and $R_1(X,Y) = fX \wedge Y$ for some function f on M_1. In this case (M,g) is conformally flat if and only if the Cotton tensor C defined by

$$C(X,Y,Z) = g((\nabla_Z L)X, Y) - g((\nabla_Y L)X, Z)$$

is zero. This implies that f is constant, i.e. (M_1, g_1) has constant sectional curvature.

Now we have to prove that if $\mathrm{Hol}^0(M,g)$ is irreducible, then it coincides with $\mathrm{SO}(r,s)$. Suppose that $\mathrm{Hol}^0(M,g)$ is irreducible and it is different from $\mathrm{SO}(r,s)$ and $\mathrm{U}(\frac{r}{2}, \frac{s}{2})$. Then the manifold is Einstein [6]. Since (M,g) is in addition conformally flat, (M,g) has constant sectional curvature and its connected holonomy group must be either trivial or $\mathrm{SO}(r,s)$, i.e. we get a contradiction. It is known that if a pseudo-Kählerian manifold is conformally flat, then it is flat [28], hence $\mathrm{Hol}^0(M,g) \neq \mathrm{U}(\frac{r}{2}, \frac{s}{2})$. This proves Theorem 5. □

3 The curvature tensor and the Weyl curvature tensor of Walker metrics

In order to prove Theorem 1, we give some information about the curvature tensor of the Walker metric (1). For the fixed coordinates $v, x^1, ..., x^n, u$ consider the field of frames

$$p = \partial_v, \quad X_i = \partial_i - A_i\partial_v, \quad q = \partial_u - \frac{1}{2}H\partial_v.$$

Consider the distribution E generated by the vector fields $X_1, ..., X_n$. The fibers of this distribution can be identified with the tangent spaces to the Riemannian manifolds with the Riemannian metrics $h(u)$. Denote by R_0 the tensor corresponding to the family of the curvature tensors of $h(u)$ under this identification. Similarly denote by $\mathrm{Ric}(h)$ the corresponding Ricci endomorphism acting on sections of E.

From the results of [14] it follows that the curvature tensor R of the metric g can be written in the form

$$R(p,q) = -\lambda p \wedge q - p \wedge \vec{v}, \qquad R(X,Y) = R_0(X,Y) - p \wedge (P(Y)X - P(X)Y), \tag{9}$$
$$R(X,q) = -g(\vec{v},X)p \wedge q + P(X) - p \wedge T(X), \qquad R(p,X) = 0 \tag{10}$$

for all $X, Y \in \Gamma(E)$. Here λ is a function, $\vec{v} \in \Gamma(E)$, $T \in \Gamma(\mathrm{End}(E))$ is symmetric, $T^* = T$, $R_0 = R(h)$, and the tensor $P \in \Gamma(E^* \otimes \mathfrak{so}(E))$ satisfies

$$g(P(X)Y,Z) + g(P(Y)Z,X) + g(P(Z)X,Y) = 0 \text{ for all } X,Y,Z \in \Gamma(E).$$

These elements may be found in terms of the coefficients of the metric (1). Let $P(X_k)X_j = P^i_{jk}X_i$ and $T(X_j) = \sum_i T_{ij}X_j$. Then

$$h_{il}P^l_{jk} = g(R(X_k,q)X_j,X_i), \quad T_{ij} = -g(R(X_i,q)q,X_j).$$

Using direct computations, we obtain

$$\lambda = \frac{1}{2}\partial_v^2 H, \quad \vec{v} = \frac{1}{2}\left(\partial_i\partial_v H - A_i\partial_v^2 H\right)h^{ij}X_j, \tag{11}$$
$$h_{il}P^l_{jk} = -\frac{1}{2}\nabla_k F_{ij} + \frac{1}{2}\nabla_k \dot{h}_{ij} - \dot{\Gamma}^l_{kj}h_{li}, \tag{12}$$
$$\begin{aligned} T_{ij} = &\frac{1}{2}\nabla_i\nabla_j H - \frac{1}{4}(F_{ik} + \dot{h}_{ik})(F_{jl} + \dot{h}_{jl})h^{kl} - \frac{1}{4}(\partial_v H)(\nabla_i A_j + \nabla_j A_i) \\ &- \frac{1}{2}(A_i\partial_j\partial_v H + A_j\partial_i\partial_v H) - \frac{1}{2}(\nabla_i\dot{A}_j + \nabla_j\dot{A}_i) \\ &+ \frac{1}{2}A_iA_j\partial_v^2 H + \frac{1}{2}\ddot{h}_{ij} + \frac{1}{4}\dot{h}_{ij}\partial_v^2 H, \end{aligned} \tag{13}$$

where

$$F = dA, \quad F_{ij} = \partial_i A_j - \partial_j A_i$$

is the differential of the 1-form A, and the covariant derivatives are taken with respect to the metric h. In the case of h, A and H independent of u, the curvature tensor of the metric (1) is found in [18].

The Ricci operator has the following form:

$$\mathrm{Ric}(p) = \lambda p, \quad \mathrm{Ric}(X) = -g(X, \widetilde{\mathrm{Ric}}\, P - \vec{v})p + \mathrm{Ric}(h)(X), \tag{14}$$
$$\mathrm{Ric}(q) = -(\mathrm{tr}\, T)p - \widetilde{\mathrm{Ric}}(P) + \vec{v} + \lambda q, \tag{15}$$

where $\widetilde{\mathrm{Ric}}\, P = h^{ij} P(X_i) X_j$ [15]. For the scalar curvature we get $s = 2\lambda + s_0$, where s_0 is the scalar curvature of h. Using this, we may compute the tensor R_L,

$$R_L(p, X) = \frac{1}{n} p \wedge \left(\mathrm{Ric}(h) + \frac{(n-1)\lambda - s_0}{n+1} \,\mathrm{id} \right) X, \tag{16}$$

$$R_L(p, q) = \frac{1}{n} \left(\frac{2n\lambda - s_0}{n+1} p \wedge q + p \wedge (\vec{v} - \widetilde{\mathrm{Ric}}\, P) \right), \tag{17}$$

$$\begin{aligned} R_L(X, Y) = \frac{1}{n} \Big(& p \wedge (g(X, \vec{v} - \widetilde{\mathrm{Ric}}\, P) Y - g(Y, \vec{v} - \widetilde{\mathrm{Ric}}\, P) X) \\ & + \left(\mathrm{Ric}(h) - \frac{s}{2(n+1)} \right) X \wedge Y + X \wedge \left(\mathrm{Ric}(h) - \frac{s}{2(n+1)} \right) Y \Big), \end{aligned} \tag{18}$$

$$\begin{aligned} R_L(X, q) = \frac{1}{n} \Big(& (\mathrm{tr}\, T) p \wedge X + g(X, \vec{v} - \widetilde{\mathrm{Ric}}\, P) p \wedge q + X \wedge (\vec{v} - \widetilde{\mathrm{Ric}}\, P) \\ & + \left(\mathrm{Ric}(h) + \frac{(n-1)\lambda - s_0}{n+1} \,\mathrm{id} \right) X \wedge q \Big). \end{aligned} \tag{19}$$

The Weyl tensor W can be computed using this and (3).

4 Proof of Theorem 1

Suppose that (M, g) is a Walker manifold, i.e. its holonomy algebra is contained in $\mathfrak{sim}(n)$. The local form of the metric g is given by (1). Suppose that g is conformally flat, i.e. $W = 0$.

Lemma 1 *The equation $W = 0$ is equivalent to the following system of equations:*

$$s_0 = -n(n-1)\lambda, \quad R_0 = -\frac{1}{2}\lambda R_{\mathrm{id}}, \quad P(X) = \vec{v} \wedge X, \quad T = f\, \mathrm{id}_E, \tag{20}$$

where X is any section of E and f is a function. In particular, $W = 0$ implies that $\widetilde{\mathrm{Ric}}\, P = -(n-1)\vec{v}$ and the Weyl tensor W_0 of h is zero.

Proof. Suppose that $W = 0$. Then it holds $R = -R_L$. From (10) and (16) it follows that $\mathrm{Ric}(h) = -\frac{(n-1)\lambda - s_0}{n+1}\, \mathrm{id}$. Taking the trace, we get $s_0 = -n(n-1)\lambda$. Hence, $\mathrm{Ric}(h) = \frac{s_0}{n}\, \mathrm{id}$, i.e. the metrics h are Einstein, and it holds

$$R_0 = W_0 - \frac{s_0}{2n(n-1)} R_{\mathrm{id}}.$$

From (9) and (18) it follows that

$$R_0 + \frac{s_0}{2n(n-1)} R_{\mathrm{id}} = 0.$$

We conclude that $W_0 = 0$. Using (10) and (19), we get $P(X) = -\frac{1}{n} X \wedge (\vec{v} - \widetilde{\mathrm{Ric}}\, P)$. Applying $\widetilde{\mathrm{Ric}}$, we obtain $\widetilde{\mathrm{Ric}}\, P = -(n-1)\vec{v}$. Consequently, $P(X) = \vec{v} \wedge X$. From (10) and (19) it follows that $T(X) = \frac{1}{n}(\mathrm{tr}\, T) X$, i.e. $T = f\, \mathrm{id}$ for a function f. Conversely, (20) implies $W = 0$. □

Since h is independent of v, $\partial_v \lambda = 0$. From (11) it follows that

$$H = \lambda v^2 + H_1 v + H_0, \quad \partial_v H_1 = \partial_v H_0 = 0.$$

From (12) it follows that the components of tensor P do not depend on the coordinate v. This, the equation $P(X) = \vec{v} \wedge X$ and (11) imply that $\partial_i \partial_v^2 H = 0$, i.e. $\partial_i \lambda = 0$, consequently s_0 and λ are functions depending only on u. Note that for $n \geq 3$ this follows also from $\mathrm{Ric}(h) = \frac{s_0}{2}\,\mathrm{id}$. We conclude that each metric in the family $h(u)$ is of constant sectional curvature. There exists a transformation $\tilde{x}^i = \tilde{x}^i(x^k, u)$ of the coordinates $x^1, ..., x^n$ depending on the parameter u such that with respect to the new coordinates the metric h takes the form

$$h = \Psi \sum_{k=1}^{n} (dx^k)^2, \quad \Psi = \frac{4}{\left(1 - \lambda(u) \sum_{k=1}^{n} (x^k)^2\right)^2}.$$

Considering the transformation $\tilde{v} = v, \tilde{x}^i = \tilde{x}^i(x^k, u), \tilde{u} = u$, we may assume that h in (1) has the just obtained form.

Let us consider the equation $T_{ij} = f\delta_{ij}$. From (13) it follows that $f = vf_1 + f_0$, $\partial_v f_1 = \partial_v f = 0$. Applying ∂_v to $T_{ij} = f\delta_{ij}$, we get the equations

$$f_1 \delta_{ij} = \frac{1}{2} \nabla_i \nabla_j H_1 - \lambda(u) \frac{1}{2} (\nabla_i A_j + \nabla_j A_i).$$

These equations are equivalent to the equations

$$\nabla_i Z_i = \nabla_j Z_j, \quad \nabla_i Z_j + \nabla_j Z_i = 0, \qquad i \neq j, \tag{21}$$

where

$$Z_i = \lambda A_i - \frac{1}{2} \partial_i H_1. \tag{22}$$

Note that $\vec{v} = -Z_i h^{ij} X_j$. The Christoffel symbols of the metric h are the following:

$$\Gamma^k_{ij} = \frac{1}{2\Psi} (\delta_{kj} \partial_i \Psi + \delta_{ki} \partial_j \Psi - \delta_{ij} \partial_k \Psi).$$

Using that, the above equations may be rewritten in the form

$$\partial_i \left(\frac{Z_i}{\Psi} \right) = \partial_j \left(\frac{Z_j}{\Psi} \right), \quad \partial_i \left(\frac{Z_j}{\Psi} \right) + \partial_j \left(\frac{Z_i}{\Psi} \right) = 0, \quad i \neq j. \tag{23}$$

We will distinguish the cases $n = 2$ and $n \geq 3$.

Case $n \geq 3$.

Lemma 2 *If $n \geq 3$, then the general solution of the system*

$$\partial_i f_i = \partial_j f_j, \quad \partial_i f_j + \partial_j f_i = 0, \quad i \neq j$$

has the form

$$f_i = x^i B_k x^k - \frac{1}{2} B_i \sum_{k=1}^{n} (x^k)^2 + d_{ik} x^k + c x^i + c_i,$$

where $B_i, c_i, c, d_{ik} \in \mathbb{R}$, $d_{ki} = -d_{ik}$.

Proof. Let i, j, k be pairwise different, then

$$\partial_i \partial_j f_k = -\partial_i \partial_k f_j = \partial_k \partial_j f_i = -\partial_i \partial_j f_k,$$

i.e. $\partial_i \partial_j f_k = 0$. This shows that $f_k = \sum_{i \neq k} C_{ki}(x^i, x^k)$. Then it is not hard to find these functions. $\square$

We conclude that

$$Z_i = \Psi \left(x^i B_k(u) x^k - \frac{1}{2} B_i(u) \sum_{k=1}^{n} (x^k)^2 + d_{ik}(u) x^k + c(u) x^i + c_i(u) \right), \tag{24}$$

where $B_i(u), c_i(u), c(u), d_{ik}(u)$ are functions of u, and $d_{ki}(u) = -d_{ik}(u)$.

The system of equations that we have solved is very similar to the equation of the Killing 1-form:

$$\nabla_i Z_j + \nabla_j Z_i = 0.$$

Let us prove the following lemma.

Lemma 3 *Any Killing vector field on the space with the metric $\Psi \sum_{k=1}^{n} (dx^k)^2$ has the following form*

$$X = X^i \partial_i, \quad X^i = f_{ik} x^k - 2\lambda x^i c_k x^k + \lambda c_i \sum_{k=1}^{n} (x^k)^2 + c_i,$$

where $b_i, f_{ik} \in \mathbb{R}$, $f_{ki} = -f_{ik}$.

Depending on the value of λ, the above metric is the metric of one of the spaces: the sphere, the Lobachevskian space, the Euclidean space. This description corresponds to the fact that the Lie algebra $\mathfrak{k}$ of the Killing vector fields on these spaces is isomorphic respectively to $\mathfrak{so}(n+1)$, $\mathfrak{so}(1,n)$, $\mathfrak{iso}(\mathbb{R}^n)$. The symmetric decomposition of the Lie algebra $\mathfrak{k}$ is of the form $\mathfrak{k} = \mathfrak{so}(n) + \mathbb{R}^n$. The vector fields defined by the numbers f_{ik} correspond to elements of $\mathfrak{so}(n)$, while the vector fields defined by the numbers c_i correspond to elements of $\mathbb{R}^n$.

Proof of Lemma 3. Consider the Killing 1-form $Z_i = h_{ij} X^j$. In addition to equations (21) it satisfies the equations $\nabla_i Z_i = 0$. These equations take the form

$$\partial_i \left(\frac{Z_i}{\Psi} \right) = -\frac{1}{2\Psi^2} \sum_{k=1}^{n} Z_k \partial_k \Psi.$$

This implies that $Z_i = \Psi f_i$, where f_i are given by Lemma 2 with $c = 0$ and $B_k = -2\lambda c_k$. This proves the Lemma. $\square$

From (22) it follows that

$$\lambda F_{ij} = \partial_i Z_j - \partial_j Z_i = \Psi^{\frac{3}{2}} \left((B_i - 2\lambda C_i) x^j - (B_j - 2\lambda C_j) x^i + 2\lambda x^k (d_{jk} x^i - d_{ik} x^j) \right) - 2\Psi d_{ij}. \tag{25}$$

This easily implies that $B_i(u) = \lambda(u) \tilde{B}_i(u)$ for some functions $\tilde{B}_i(u)$.

From (12) and (11) it follows that the equation $P(X) = \vec{v} \wedge X$ takes the form

$$-\frac{1}{2} \nabla_k F_{ij} - \delta_{ik} \frac{\Psi}{2} \partial_u \partial_j \ln \Psi + \delta_{jk} \frac{\Psi}{2} \partial_u \partial_j \ln \Psi$$
$$= \Psi \left(\frac{1}{2} \partial_j H_1 - \lambda A_j \right) \delta_{ki} - \Psi \left(\frac{1}{2} \partial_i H_1 - \lambda A_i \right) \delta_{kj}. \tag{26}$$

If i, j, k are pair-wise different, then

$$0 = \nabla_k F_{ij} = \partial_k F_{ij} - \frac{1}{\Psi} F_{ij} \partial_k \Psi - \frac{1}{2\Psi} F_{kj} \partial_i \Psi - \frac{1}{2\Psi} F_{ik} \partial_j \Psi$$
$$= \Psi \partial_k \left(\frac{F_{ij}}{\Psi} \right) - \lambda F_{kj} x^i \sqrt{\Psi} - \lambda F_{ik} x^j \sqrt{\Psi}.$$

Using this and Equation (25), we obtain

$$-\partial_k\left(\frac{F_{ij}}{\Psi}\right)=\lambda x^k\Psi\big((\tilde B_j-2C_j)x^i-(\tilde B_i-2C_i)x^j\big)+2\sqrt{\Psi}(x^id_{kj}-x^jd_{ki})\big)+2\lambda\Psi x^k(x^jd_{il}-x^id_{jl})x^l.$$

Integrating over x^k, we get

$$F_{ij}=\Psi^{\frac{3}{2}}\big((\tilde B_i-2C_i)x^j-(\tilde B_j-2C_j)x^i+2(d_{li}x^j-d_{lj}x^i)x^l\big)-\Psi C_{ij}(x^i,x^j,u) \qquad (27)$$

for some functions $C_{ij}(x^i,x^j,u)$. Comparing this with (25), we conclude that $d_{ij}=\frac{\lambda}{2}C_{ij}$. Equation (26) for $k=i\neq j$ is of the form

$$\nabla_iF_{ij}=-\Psi\partial_u\partial_j\ln\Psi+2\Psi Z_j.$$

Direct computations show that

$$\nabla_iF_{ij}=\Psi^{\frac{3}{2}}\partial_i\left(\frac{F_{ij}}{\Psi^{\frac{3}{2}}}\right)+\lambda F_{lj}x^l\sqrt{\Psi}.$$

Using this, (22), (25) and (27), we get

$$\Psi^{\frac{3}{2}}\left(-2d_{ij}-\partial_i\frac{C_{ij}}{\sqrt{\Psi}}\right)+\Psi\partial_u\partial_j\ln\Psi-\Psi^2\left(\lambda D_kx^kx^j-\frac{1}{2}D_j\sum_{k=1}^n(x^k)^2+2c(u)x^j+\frac{1}{2}D_j\right)=0,$$

where $D_k=\tilde B_k+2C_k$. Using the equalities

$$\partial_j\partial_u\ln\Psi=\dot\lambda x^j\Psi,\quad \partial_i\frac{C_{ij}}{\sqrt{\Psi}}=-\lambda x_iC_{ij}+\frac{\partial_iC_{ij}}{\sqrt{\Psi}},\quad d_{ij}=\frac{\lambda}{2}C_{ij},$$

we get

$$-\partial_iC_{ij}-\Psi\left(\lambda D_kx^kx^j-\frac{1}{2}D_j\sum_{k=1}^n(x^k)^2+(2c(u)-\dot\lambda)x^j+\frac{1}{2}D_j\right)=0.$$

If $\lambda(u)\neq0$, then the equality $d_{ij}=\frac{\lambda}{2}C_{ij}$ implies $\partial_iC_{ij}=0$. Consequently, $D_k=0$ and $c=\frac{1}{2}\dot\lambda$. If $\lambda(u)=0$, then we get

$$\partial_j\partial_iC_{ij}+4(D_jx_j-(2c-\dot\lambda))=0.$$

Here we wrote x_j instead of x^j in order to avoid the sum over j. Since $C_{ij}=-C_{ji}$, we get $D_j=0$, $c=\frac{1}{2}\dot\lambda$, and $\partial_iC_{ij}=0$. Thus it holds $D_k=0$, $\tilde B_k=-2C_k$, $c=\frac{1}{2}\dot\lambda$, and $d_{ij}=\frac{\lambda}{2}C_{ij}$, where C_{ij} are functions of u. We conclude that

$$F_{ij}=\Psi^{\frac{3}{2}}\big(4C_j(u)x^i-4C_i(u)x^j+\lambda(u)(C_{li}(u)x^j-C_{lj}(u)x^i)x^l\big)-\Psi C_{ij}(u). \qquad (28)$$

Recall that $F=dA$. In [18] it is noted that the transformation $v\mapsto v-\phi(x^1,...,x^m,u)$ changes A_i to $A_i+\partial_i\phi$. Clearly, this transformation does not change F. This allows us to choose any A such that $dA=F$. We take

$$A_i=\Psi\left(-4C_k(u)x^kx^i+2C_i(u)\sum_{k=1}^n(x^k)^2+\frac{1}{2}C_{ik}(u)x^k\right).$$

Consider the coordinate transformation with the inverse one

$$v=\tilde v,\quad x^i=A^i_j(\tilde u)\tilde x^j,\quad u=\tilde u, \qquad (29)$$

where $A^i_j(u)$ is a family of orthogonal matrices. It is easy to check that

$$\tilde{A}_i = \sum_{k=1}^{n} A^k_i(u)(\partial_u A^k_l(u))\tilde{x}^l + A^k_i(u)A_i.$$

The obtained metric has the same form and it holds

$$\tilde{C}_i(u) = C_j(u)A^j_i(u), \quad \tilde{C}_{ij}(u) = \sum_{k=1}^{n} A^k_i(u)\partial_u A^k_j(u) + \frac{1}{2}A^r_i(u)C_{rk}(u)A^k_j(u).$$

Consider the equation $\tilde{C}_{ij}(u) = 0$. Since $\sum_{k=1}^{n} A^i_k(u)A^j_k(u) = \delta^{ij}$, it can be written in the form

$$\partial_u A^k_i(u) = A^j_i(u)\frac{1}{2}C_{jk}(u).$$

Since $C_{jk}(u)$ is skew-symmetric, $\frac{1}{2}C_{jk}(u)$ is a curve in the Lie algebra $\mathfrak{so}(n)$. Then $A^k_i(u)$ satisfying the above equation is nothing else as the development of the curve $\frac{1}{2}C_{jk}(u)$ in the Lie group $\mathrm{SO}(n)$. Thus, applying such transformation, we may assume that $C_{ij}(u) = 0$.

Next,

$$\partial_i H_1 = 2\lambda A_i - 2Z_i = -2\Psi\left(2\lambda C_k(u)x^k x^i - \lambda C_i(u)\sum_{k=1}^{n}(x^k)^2 + \frac{1}{2}\dot{\lambda}x^i + C_i(u)\right).$$

We conclude that

$$H_1 = -4C_k(u)x^k\sqrt{\Psi} - \partial_u \ln\Psi + K(u)$$

for some function $K(u)$.

We are left with the only unknown function H_0. Consider equations $T_{ij} = f\delta_{ij}$. If $i \neq j$, then $\nabla_i\nabla_j H_0 = \sqrt{\Psi}\partial_i\partial_j\frac{H_0}{\sqrt{\Psi}}$, and using (13), we obtain

$$\partial_i\partial_j\frac{H_0}{\sqrt{\Psi}} = 2\Psi^{\frac{3}{2}}x^i x^j\sum_{k=1}^{n} C_k^2(u).$$

If $\lambda(u) \neq 0$, then the function $\frac{4}{\lambda^2(u)}\Psi\sum_{k=1}^{n} C_k^2(u)$ is a partial solution of the system. The general solution is of the form

$$H_0 = \frac{4}{\lambda^2(u)}\Psi\sum_{k=1}^{n} C_k^2(u) + \sqrt{\Psi}\sum_{k=1}^{n} f_k(x^k, u).$$

The condition $\nabla_i\nabla_i H_0 = \nabla_j\nabla_j H_0$ implies $\partial_i^2 f_i = \partial_j^2 f_j$, hence

$$f_i(x^i, u) = a(u)(x^i)^2 + D_i(u)x_i + d_i(u)$$

for some functions $a(u)$, $D_i(u)$, $d_i(u)$. We see that H_0 is as in the statement of the theorem for the case $\lambda(u) \neq 0$. The case $\lambda(u) = 0$ is similar.

Case $n = 2$. The system of equations (23) takes the form

$$\partial_1\left(\frac{Z_1}{\Psi}\right) = \partial_2\left(\frac{Z_2}{\Psi}\right), \quad \partial_1\left(\frac{Z_2}{\Psi}\right) + \partial_2\left(\frac{Z_1}{\Psi}\right) = 0 \tag{30}$$

that implies only that $\frac{Z_1}{\Psi}$ and $\frac{Z_2}{\Psi}$ are real and imaginary parts of a complex homomorphic function of the variable $x^1 + ix^2$.

Note that

$$\nabla_1 F_{12} = \Psi\partial_1\frac{F_{12}}{\Psi}, \quad \nabla_2 F_{12} = \Psi\partial_2\frac{F_{12}}{\Psi}.$$

Using (26), we get

$$Z_1 = -\frac{1}{2}\partial_2\frac{F_{12}}{\Psi} + \frac{1}{2}\partial_u\partial_1 \ln\Psi, \quad Z_2 = \frac{1}{2}\partial_1\frac{F_{12}}{\Psi} + \frac{1}{2}\partial_u\partial_2 \ln\Psi.$$

Substituting that to the first equation in (30), and using the equality $\partial_u\partial_i \ln\Psi = \dot{\lambda}x^i\Psi$, we get

$$\partial_1\partial_2\frac{F_{12}}{\Psi^{\frac{3}{2}}} = 0.$$

This implies

$$F_{12} = \Psi^{\frac{3}{2}}(f_1(x^1, u) + f_2(x^2, u)).$$

Substituting that to the second equation in (30), we obtain $\partial_1^2 f_1 = \partial_2^2 f_2$, i.e.

$$f_1 = t(u)(x^1)^2 + a^1(u)x^1 + b_1(u), \quad f_2 = t(u)(x^2)^2 + a^2(u)x^2 + b_2(u).$$

It holds

$$\partial_1 H_1 = 2Z_1 + 2\lambda A_1, \quad \partial_2 H_1 = 2Z_2 + 2\lambda A_2.$$

Consequently,

$$0 = \partial_2\partial_1 H_1 - \partial_1\partial_2 H_1 = -2\lambda F_{12} + 2\partial_2 Z_1 - 2\partial_1 Z_2.$$

Direct computations show that this implies $t = \lambda b$. Thus,

$$F_{12} = \Psi^{\frac{3}{2}}(\lambda(u)b(u)((x^1)^2 + (x^2)^2) + a_i(u)x^i(u) + b(u)).$$

Equality (28) for $n = 2$ can be rewritten in the form

$$F_{12} = \Psi^{\frac{3}{2}}\left(-\frac{1}{2}\lambda(u)C_{12}(u)((x^1)^2 + (x^2)^2) - \frac{1}{2}C_{12}(u) + 4C_2(u)x^1 - 4C_1(u)x^2\right).$$

This shows that F_{ij} is the same as for $n \geq 3$, and the rest of the proof for $n = 2$ is the same as for $n \geq 3$. The theorem is true. □

5 Proof of Theorem 2

Consider the coordinates $v, x^1, ..., x^n, u$ and the metric g is in Theorem 1.

1) Suppose that $\lambda(u)$ is nowhere vanishing. Consider the transformation

$$v \mapsto v - \phi, \quad \phi = \frac{2}{\lambda(u)}C_k(u)x^k\sqrt{\Psi}.$$

Then A_i changes to

$$A_i + \partial_i\phi = \Psi\left(-2C_k(u)x^k x^i + C_i(u)\sum_{k=1}^{n}(x^k)^2 + \frac{C_i(u)}{\lambda(u)}\right).$$

By Lemma 3, for each u, $h^{ik}A_k$ is a Killing vector field of the Riemannian metric $h(u)$.

Following the ideas from [16], we are looking for a coordinate transformation in order to set A_i to zero. Consider the coordinate transformation with the inverse one given by

$$v = \tilde{v},\ x^k = x^k(\tilde{x}^i, u),\ u = \tilde{u}.$$

It holds

$$\tilde{A}_k = \frac{\partial x^i}{\partial \tilde{x}^k}\left(A_i + h_{ij}\frac{\partial x^j}{\partial u}\right).$$

The equation $\tilde{A}_k = 0$ is of the form

$$\frac{\partial x^j(\tilde{x}^1, ..., \tilde{x}^n, u)}{\partial u} = V^j(x^1(\tilde{x}^1, ..., \tilde{x}^n, u), ..., x^n(\tilde{x}^1, ..., \tilde{x}^n, u), u),$$

where $V^j = -h^{jk}A_k$. Considering $\tilde{x}^1, ..., \tilde{x}^k$ as parameters and imposing the initial dates

$$x^i(\tilde{x}^1, ..., \tilde{x}^n, u_0) = \tilde{x}^i,$$

we see that the above system is a system of ordinary differential equations depending on the initial dates as on parameters. Such system has a unique solution that gives the required transformation. Since for each u, $h^{ik}A_k$ is a Killing vector field of the Riemannian metric $h(u)$, the transformation $x^k = x^k(\tilde{x}^i, u)$ is an isometry, hence h remains the same. Thus we get the metric from Theorem 1 with $C_i(u) = 0$. Applying the transformation $v \mapsto v + \frac{1}{2\lambda(u)}K(u)$, we get $H_1 = -\partial_u \ln \Psi$.

2) Suppose that λ is constantly zero. The from F is given by

$$F_{ij} = 32(C_j(u)x^i - C_i(u)x^j).$$

As it is explained in the proof of Theorem 1, we may take $A_i = 16C_i(u)\sum_{k=1}^n (x^k)^2$. Considering the transformation $x^i \mapsto \frac{x^i}{2}$, and redenoting $2C_i(u)$ by $C_i(u)$, we get $A = C_i(u)\sum_{k=1}^n (x^k)^2 dx^i du$, i.e. $A_i = C_i(u)\sum_{k=1}^n (x^k)^2$. Moreover, $h = \sum_{k=1}^n (dx^k)^2$. From the equation $P(X) = \vec{v} \wedge X$ it follows that

$$\partial_j H_1 = -\nabla_i F_{ij} = -2C_j.$$

Hence, $H_1 = -2C_k x^k + K(u)$ for some function $K(u)$.

Suppose that $\sum_{k=1}^n C_k^2 \not\equiv 0$. Consider the coordinate transformation with the inverse one

$$v = \tilde{v}, \quad x^i = \tilde{x}^i + b^i(\tilde{u}), \quad u = \tilde{u}$$

such that $2C_i(u)b^i(u) = K(u)$. After that $H_1 = -2C_i(u)x^i$, i.e. we may assume that $K(u) = 0$. The equation $T_{ij} = f\delta_{ij}$ for $i \neq j$ takes the form

$$\frac{1}{2}\partial_i\partial_j H_0 = x^i x^j \sum_{k=1}^n C_k - 2C_k x^k (C_j x^i + C_i x^j) - C_i C_j \sum_{k=1}^n (x^k)^2 + (\dot{C}_j x^i + \dot{C}_i x^j).$$

The general solution to this system under the condition $T_{ii} = T_{jj}$ is given in the statement of the theorem.

Suppose that $\sum_{k=1}^n C_k^2 \equiv 0$. It is easy to check that the non-zero components of the curvature tensor are R_{iuju}, R_{uiuj}, R_{iuuj}, R_{uiju}, hence the metric is a pp-wave [13], i.e. $h = \sum_{k=1}^n (dx^k)^2$, $A = 0$, and $H = H_0$. Moreover,

$$H_0 = a(u)\sum_{i=1}^n (x^i)^2 + D_i(u)x^i + D(u).$$

Consider the new coordinates

$$\tilde{v} = v - \sum_j \frac{db^j(u)}{du} x^j + d(u), \quad \tilde{x}^i = x^i + b^i(u), \quad \tilde{u} = u.$$

We obtain the metric of the same form with

$$\tilde{H}_0 = a(u) \sum_{i=1}^n (x^i)^2 + \tilde{D}_i(u) x^i + \tilde{D}(u),$$

where

$$\tilde{D}_j = -2\frac{d^2 b^j}{(du)^2} + 2ab^j + D_j, \tag{31}$$

$$\tilde{D} = 2\frac{dd(u)}{du} + \sum_j \left(\frac{db^j}{du}\right)^2 + a\sum_j (b^j)^2 + D_i b^i + D. \tag{32}$$

Equation (31) implies the existence of $b^j(u)$ such that $\tilde{D}_k = 0$. Using the last equation, we can chose $d(u)$ such that $\tilde{D} = 0$. Thus,

$$H_0 = a(u) \sum_{i=1}^n (x^i)^2.$$

The theorem is true. $\square$

6 Proof of Theorem 3

We consider the metric g from Theorem 1 and consider different cases.

Suppose that $\lambda \equiv 0$ on M. Suppose that $\sum_{k=1}^n C_k^2 \not\equiv 0$. Then there exists a point $x \in M$ such that $\vec{v}_x \neq 0$. The condition on the curvature tensor shows that

$$R_x(p_x, q_x) = -p_x \wedge \vec{v}_x, \quad R_x(X, Y) = p_x \wedge ((X \wedge Y)\vec{v}_x).$$

This shows that $p_x \wedge E_x \subset \mathfrak{g}$. Next,

$$R_x(\vec{v}_x, q_x) = -g(\vec{v}_x, \vec{v}_x) p_x \wedge q_x - p_x \wedge T_x(\vec{v}_x),$$

which implies $\mathbb{R} p_x \wedge q_x \subset \mathfrak{g}$. Finally,

$$R_x(X, q_x) = -g(\vec{v}_x, X) p_x \wedge q_x + \vec{v}_x \wedge X - p_x \wedge T_x(X).$$

Since the bivectors of the form $\vec{v}_x \wedge X$ generate the Lie algebra $\mathfrak{so}(E_x)$, we conclude that

$$\mathfrak{g} = \mathbb{R} p_x \wedge q_x + \mathfrak{so}(E_x) + p_x \wedge E_x \simeq \mathfrak{sim}(n).$$

If $\sum_{k=1}^n C_k^2 \equiv 0$ and $a^2 \not\equiv 0$, then the metric is given by (2), and its holonomy algebra coincides with $\mathbb{R}^n$. If $\sum_{k=1}^n C_k^2(u) \equiv 0$ and $a^2(u) \equiv 0$, then the metric is flat.

Suppose that λ is a non-zero constant, then the metric can be written as the first metric from Theorem 2. It holds

$$R(p, q) = -\lambda p \wedge q, \quad R(X, Y) = -\frac{1}{2}\lambda X \wedge Y, \quad X, Y \in \Gamma(E).$$

This shows that $\mathfrak{g}$ contains the subalgebra $\mathbb{R}\oplus\mathfrak{so}(n)\subset\mathfrak{sim}(n)$. Next,

$$T_{ii}=\lambda\Psi^{\frac{3}{2}}D_kx^k+\Psi(\sqrt{\Psi}-1)(a+\lambda D).$$

If $\sum_{k=1}^n D_k^2+(a+\lambda D)^2\not\equiv 0$, then the tensor T is not identical zero, and the equality $R(X,q)=-n\operatorname{tr}Tp\wedge X$ shows that $\mathfrak{g}$ contains $\mathbb{R}^n\subset\mathfrak{sim}(n)$, i.e. $\mathfrak{g}=\mathfrak{sim}(n)$. Otherwise,

$$g=\Psi\sum_{k=1}^n(dx^k)^n+2dvdu+(\lambda v^2+2D(u))(du)^2.$$

We see that g is decomposable. The metric $2dvdu+(\lambda v^2+2D(u))(du)^2$ is of constant sectional curvature $\frac{\lambda}{2}$, hence it is isometric to the metric $2dvdu+\lambda v^2(du)^2$.

Suppose that $\dot\lambda\not\equiv 0$ for some coordinate system. Then $\lambda\not\equiv 0$ on an open subset of the definition domain of this system, and the metric can be written as the first metric from Theorem 2. It holds $\vec v=-\frac{1}{2}\dot\lambda x^i\partial_i$. Let x be any point such that $\vec v_x\neq 0$. It holds

$$R_x(p_x,q_x)=-\lambda_xp_x\wedge q_x-p_x\wedge\vec v_x\in\mathfrak{g},$$

$$R_x(X,Y)=-\frac{\lambda}{2}X\wedge Y+p_x\wedge((Y\wedge X)\vec v_x)\in\mathfrak{g},\quad X,Y\in T_xM.$$

Taking the Lie bracket of the last two elements of $\mathfrak{g}$, we obtain $p_x\wedge((Y\wedge X)\vec v_x)\in\mathfrak{g}$. This shows that $\mathfrak{g}$ contains the subalgebra isomorphic to $\mathbb{R}^n\subset\mathfrak{g}$. The above two equalities imply that $\mathfrak{g}$ contains the subalgebra $\mathbb{R}\oplus\mathfrak{so}(n)\subset\mathfrak{sim}(n)$. Thus, $\mathfrak{g}=\mathfrak{sim}(n)$.

The theorem is true. $\square$

7 The Ricci operator of the obtained metrics

The Ricci operator of the metric (2) has the form

$$\mathrm{Ric}=na(u)\partial_v\otimes du,$$

in particular, $\mathrm{Ric}^2=0$.

In [7], complete conformally flat Lorentzian manifolds (M,g) satisfying the condition

$$[R(X,Y),\mathrm{Ric}]=0\tag{33}$$

are studied. It is shown that these manifolds are exhausted by the spaces of constant sectional curvature, by the products of two spaces of constant sectional curvature, and by products of spaces of constant sectional curvature with intervals. The Ricci operator of the metric (2) satisfies (33). Moreover, such metric is complete, e.g. for $a(u)=1$, i.e. for the Cahen-Wallach spaces. Thus some metrics in [7] are loosen. In [10], pseudo-Riemannian conformally flat manifolds (M,g) satisfying (33) are studied. It is shown that in addition to the obvious cases, (M,g) may be a complex sphere or a space satisfying $\mathrm{Ric}^2=0$. Various examples of conformally flat manifolds with $\mathrm{Ric}^2=0$ are constructed in [11].

The Ricci operator of the second metric from Theorem 2 is the following:

$$\mathrm{Ric}(p)=0,\quad\mathrm{Ric}(X)=ng(X,\vec v)p,\quad\mathrm{Ric}(q)=nT_{11}p+n\vec v,\quad X\in\Gamma(E),$$

here $\vec v=-\sum_{k=1}^n C_k(u)X_k$. The function T_{11} can be found using (13). It holds $\mathrm{Ric}^2\neq 0$ and $\mathrm{Ric}^3=0$. Condition (33) is not satisfied. The scalar curvature of this metric is zero.

For the first metric form Theorem 2 it holds

$$\mathrm{Ric}(p)=\lambda p,\quad\mathrm{Ric}(X)=ng(X,\vec v)p-(n-1)\lambda X,\quad\mathrm{Ric}(q)=nT_{11}p+n\vec v+\lambda q,$$

where $\vec v=-\frac{1}{2}\dot\lambda\Psi x^kX_k$. The Ricci operator is not nilpotent. The scalar curvature equals to $2\lambda+s_0=-(n-2)(n+1)\lambda$ and it is zero only in dimension four.

8 The case of dimension 4

Applying Theorem 5 to a conformally flat nonflat Lorentzian manifold (M, g) of dimension 4 with the holonomy algebra $\mathfrak{g} \subset \mathfrak{so}(1, 3)$, we obtain that (M, g) must satisfy one of the following conditions:

1) $\mathfrak{g} = \mathfrak{so}(1, 3)$;

2) $\mathfrak{g} \subset \mathfrak{sim}(2)$, i.e. (M, g) is as in Theorem 1 with $n = 2$;

3) $\mathfrak{g} = \mathfrak{so}(1, 1) \oplus \mathfrak{so}(2)$, and (M, g) is locally isometric either to the product of (dS_2, cg_{dS_2}) with (L^2, cg_{L^2}), or to the product of (AdS_2, cg_{AdS_2}) with (S^2, cg_{S^2});

4) $\mathfrak{g} = \mathfrak{so}(1, 2)$, and (M, g) is locally isometric to the product of $(\mathbb{R}, (dt)^2)$ either with (dS_3, cg_{dS_3}), or with (AdS_3, cg_{AdS_3}); or $\mathfrak{g} = \mathfrak{so}(3)$, and (M, g) is locally isometric to the product of $(\mathbb{R}, -(dt)^2)$ either with (S^3, cg_{S^3}), or with (L^3, cg_{L^3}).

Here $c > 0$ is a constant, and S^n, L^n, dS_n, AdS_n denote, respectively the sphere, Lobachevskian space, de Sitter space and anti de Sitter space with there standard metrics. The standard Friedmann-Robertson-Walker spacetimes are conformally flat and give examples of holonomy $\mathfrak{so}(1, 3)$ [9].

Possible holonomy algebras of conformally flat 4-dimensional Lorentzian manifolds are classified also in [9]. The first metric from Theorem 1 in dimension 4 is given in [25]. In [9], it is stated that it is an open problem to construct a conformally flat metric with the holonomy algebra $\mathfrak{sim}(2)$ (which is denoted in [9] by R_{14}). An attempt to construct such metric is made in [17], where the following metric is constructed:

$$g = 2dxdt + 4ydxdy - 4zdxdz + \frac{(dy)^2}{2y^2} + \frac{(dz)^2}{2y^2} + 2(x + y^2 - z^2)^2(dt)^2. \quad (34)$$

Using Maple, it is easy to check that the Weyl tensor of this metric is not zero, i.e. this metric is not conformally flat. Next, the authors were looking for a metric written in Walker coordinates. These two arguments suggest that the metric must reed as follows:

$$g = 2dxdt + 4ydtdy - 4zdtdz + \frac{(dy)^2}{2y^2} + \frac{(dz)^2}{2y^2} + 2(x + y^2 - z^2)^2(dt)^2. \quad (35)$$

This metric is conformally flat. Making the transformation

$$x \mapsto x - y^2 + z^2, \quad y \mapsto y, \quad z \mapsto z, \quad t \mapsto t,$$

we obtain

$$g = 2dxdt + 2x^2(dt)^2 + \frac{(dy)^2}{2y^2} + \frac{(dz)^2}{2y^2}. \quad (36)$$

We get that the metric (35) is decomposible and its holonomy algebra coincides with $\mathfrak{so}(1, 1) \oplus \mathfrak{so}(2)$, but not with $\mathfrak{sim}(2)$. Thus in this paper we get metrics with the holonomy algebra $\mathfrak{sim}(2)$ for the first time (even more, recall that we find all such metrics).

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