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Conformally flat Lorentzian manifolds with special holonomy groups

Anton S. Galaev

Masaryk University (Brno, Czech Republic)

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Theorem (Kurita 1955). Let (M, g) be an n-dimensional conformally flat Riemannian manifold. Then its local restricted holonomy group H_x $(x \in M)$ is in general $\mathrm{SO}(n)$. If $H_x \neq \mathrm{SO}(n)$, then for some coordinate neighborhood U of x one of the following holds:

1) H_x is identity and the metric is flat in U;
2) $H_x = \mathrm{SO}(k) \times \mathrm{SO}(n-k)$ and U is a direct product of a k-dimensional manifold of constant sectional curvature K and an $(n-k)$-dimensional manifold of constant sectional curvature $-K$ $(K \neq 0)$;
3) $H_x = \mathrm{SO}(n-1)$ and U is a direct product of a straight line (or a segment) and an $(n-1)$-dimensional manifold of constant sectional curvature.

One says that the connected holonomy group of an indecomposable pseudo-Riemannian manifold is **special** if it is different from the connected component of the pseudo-orthogonal group.

There are no conformally flat Riemannian manifolds with special holonomy groups.

A subgroup $G \subset \mathrm{SO}(r,s)$ (a subalgebra $\mathfrak{g} \subset \mathfrak{so}(r,s)$) is called **weakly irreducible** if it does not preserve any non-degenerate proper vector subspace of the tangent space.

A pseudo-Riemannian manifold is not locally decomposable iff its connected holonomy group is weakly irreducible.

Theorem. Let (M,g) be a conformally flat pseudo-Riemannian manifold of signature (r,s) with the restricted holonomy group $\mathrm{Hol}^0(M,g)$. If (M,g) is not flat, then one of the following holds:

1) $\mathrm{Hol}^0(M,g)=\mathrm{SO}(r,s)$;
2) $\mathrm{Hol}^0(M,g)$ is weakly irreducible and not irreducible;
3) $\mathrm{Hol}^0(M,g)=\mathrm{SO}(r_1,s_1)\times\mathrm{SO}(r-r_1,s-s_1)$ and (M,g) is locally a product of a pseudo-Riemannian manifold of constant sectional curvature K and signature (r_1,s_1) and a pseudo-Riemannian manifold of constant sectional curvature $-K$ $(K\neq 0)$ and signature $(r-r_1,s-s_1)$;
4) $\mathrm{Hol}^0(M,g)=\mathrm{SO}(r-1,s)$ (resp., $\mathrm{Hol}^0(M,g)=\mathrm{SO}(r,s-1)$) and (M,g) is locally a product of a pseudo-Riemannian manifold of constant sectional curvature and signature $(r-1,s)$ (resp., $(r,s-1)$) and the space $(L,-(dt)^2)$ (resp.,

Lorentzian holonomy algebras.

Let (M,g) be a locally indecomposable Lorentzian manifold of dimension $n+2\geq 4$ and $\mathfrak{g}\subset\mathfrak{so}(1,n+1)$ be its holonomy algebra, which is weakly irreducible.

If $\mathfrak{g}\subset\mathfrak{so}(1,n+1)$ is irreducible, then $\mathfrak{g}=\mathfrak{so}(1,n+1)$.

Any weakly irreducible holonomy algebra $\mathfrak{g}\subsetneq\mathfrak{so}(1,n+1)$ preserves an isotropic line of the tangent space $\mathbb{R}^{1,n+1}$.

Fix two isotropic vectors $p, q \in \mathbb{R}^{1,n+1}$ such that $g(p,q) = 1$. Let $E \subset \mathbb{R}^{1,n+1}$ be the orthogonal complement to $\mathbb{R}p \oplus \mathbb{R}q$. Then

$$\mathbb{R}^{1,n+1} = \mathbb{R}p \oplus E \oplus \mathbb{R}q.$$

Denote by $\mathfrak{sim}(n)$ the maximal subalgebra of $\mathfrak{so}(1, n+1)$ preserving $\mathbb{R}p$.

In the matrix form:

$$\mathfrak{sim}(n) = \left\{ \left(\begin{array}{ccc} a & X^t & 0 \\ 0 & A & -X \\ 0 & 0 & -a \end{array} \right) \middle| \begin{array}{c} a \in \mathbb{R}, \\ X \in \mathbb{R}^n, \\ A \in \mathfrak{so}(n) \end{array} \right\}.$$

Identify $\mathfrak{so}(1,n+1)$ with $\Lambda^2\mathbb{R}^{1,n+1}$ in such a way that

$$(X\wedge Y)Z=(X,Z)Y-(Y,Z)X,$$

then

$$\mathfrak{sim}(n)=(\mathbb{R}\oplus\mathfrak{so}(n))+\mathbb{R}^n$$

$$=\mathbb{R}p\wedge q+\mathfrak{so}(E)+p\wedge E.$$

Theorem. (Berard-Bergery, Ikemakhen, Leistner, Galaev)

The Lorentzian holonomy algebras $\mathfrak{g} \subset \mathfrak{sim}(n)$ are the following :

(type I) $\mathbb{R}p \wedge q + \mathfrak{h} + p \wedge E$,

(type II) $\mathfrak{h} + p \wedge E$,

(type III) $\{\varphi(A)p \wedge q + A | A \in \mathfrak{h}\} + p \wedge E$,

(type IV) $\{A + p \wedge \psi(A) | A \in \mathfrak{h}\} + p \wedge E_1$,

where $\mathfrak{h} \subset \mathfrak{so}(E)$ is a Riemannian holonomy algebra; $\varphi : \mathfrak{h} \to \mathbb{R}$ is a non-zero linear map, $\varphi|_{[\mathfrak{h},\mathfrak{h}]} = 0$; for the last algebra $E = E_1 \oplus E_2$, $\mathfrak{h} \subset \mathfrak{so}(E_1)$, and $\psi : \mathfrak{h} \to E_2$ is a surjective linear $\psi|_{[\mathfrak{h},\mathfrak{h}]}$.

Let (M, g) be a Lorentzian manifold with the holonomy algebra $\mathfrak{g} \subset \mathfrak{sim}(n)$.

Locally there exist so called Walker coordinates $v, x^1, ..., x^n, u$ such that the metric g has the form

$$g = 2dvdu + h + 2Adu + H(du)^2, \qquad (2.1)$$

where

$h = h_{ij}(x^1, ..., x^n, u)dx^i dx^j$ is an u-family of Riemannian metrics,
$A = A_i(x^1, \dots, x^n, u)dx^i$ is an u-family of one-forms,
$H = H(v, x^1, ..., x^n, u)$ is a local function on M.

On a Walker manifold (M,g) we define the canonical function λ from the equality

$$\mathrm{Ric}\, p = \lambda p,$$

$\lambda = \frac{1}{2}\partial_v^2 H,$

The scalar curvature of g:

$s = 2\lambda + s_0$, where s_0 is the scalar curvature of h.

The Main Theorem. Let (M,g) be a conformally flat Walker Lorentzian manifold. Then locally

$$g=2dvdu+\Psi\sum_{i=1}^{n}(dx^i)^2+2Adu+(\lambda(u)v^2+vH_1+H_0)(du)^2,$$

where

$$\Psi=\frac{4}{(1-\lambda(u)\sum_{k=1}^{n}(x^k)^2)^2},$$

$$A=A_idx^i,\quad A_i=\Psi\left(-4C_k(u)x^kx^i+2C_i(u)\sum_{k=1}^{n}(x^k)^2\right),$$

$$H_1=-4C_k(u)x^k\sqrt{\Psi}-\partial_u\ln\Psi+K(u),$$

$$s=-(n-2)(n+1)\lambda(u)$$

Theorem.

If the function λ is non-vanishing at a point, then in a neighborhood of this point there exist coordinates $v, x^1, ..., x^n, u$ such that

$$g = 2dvdu + \Psi \sum_{i=1}^{n} (dx^i)^2 + (\lambda(u)v^2 + vH_1 + H_0)(du)^2,$$

where

$$\Psi = \frac{4}{(1 - \lambda(u)\sum_{k=1}^{n}(x^k)^2)^2},$$

$$H_1 = -\partial_u \ln \Psi, \quad H_0 = \sqrt{\Psi}\left(a(u)\sum_{k=1}^{n}(x^k)^2 + D_k(u)x^k + D(u)\right).$$

Theorem. If $\lambda \equiv 0$ in a neighborhood of a point, then in a neighborhood of this point there exist coordinates $v, x^1, ..., x^n, u$ such that

$$g = 2dvdu + \sum_{i=1}^{n}(dx^i)^2 + 2Adu + (vH_1 + H_0)(du)^2,$$

where

$$A = A_i dx^i, \quad A_i = C_i(u)\sum_{k=1}^{n}(x^k)^2, \quad H_1 = -2C_k(u)x^k$$

$$H_0 = \sum_{k=1}^{n}(x^k)^2\left(\frac{1}{4}\sum_{k=1}^{n}(x^k)^2\sum_{k=1}^{n} C_k^2(u) - (C_k(u)x^k)^2 + \dot{C}_k(u)x^k + a(u)\right)$$

In particular, if all $C_i \equiv 0$, then the metric can be rewritten in the form

$$g = 2dvdu + \sum_{i=1}^{n}(dx^i)^2 + a(u)\sum_{k=1}^{n}(x^k)^2(du)^2. \qquad (2.2)$$

Remarks.

The field equations of Nordström's theory of gravitation, which appeared before Einstein's theory, are the following:

$$W = 0, \quad s = 0.$$

Thus we have found all solutions to Nordström's gravity with holonomy algebras contained in $\mathfrak{sim}(n)$.

Similarly, the Einstein equation on Lorentzian manifolds with such holonomy algebras was studied in

G. W. Gibbons, C. N. Pope, *Time-Dependent Multi-Centre Solutions from New Metrics with Holonomy* $\mathrm{Sim}(n-2)$, Class. Quantum Grav. 25 (2008) 125015 (21pp).

In this case it is impossible to obtain the complete solution, but the examples of solutions have interesting physical interpretations

The case of dimension 4.

Possible holonomy algebras of conformally flat 4-dimensional Lorentzian manifolds were classified also in

G. S. Hall, D. P. Lonie, *Holonomy groups and spacetimes*, Class. Quantum Grav. 17 (2000), 1369–1382.

It is stated that it is an open problem to construct a conformally flat metric with the holonomy algebra $\mathfrak{sim}(2)$ (which is denoted in by R_{14}).

An attempt to construct such metric is made in

R. Ghanam, G. Thompson, *Two special metrics with R_{14}-type holonomy*, Class. Quantum Grav. 18 (2001), 2007–2014

where the following metric was constructed:

$$g = 2dxdt + 4ydtdy - 4zdtdz + \frac{(dy)^2}{2y^2} + \frac{(dz)^2}{2y^2} + 2(x + y^2 - z^2)^2(dt)^2.$$

Making the transformation

$$x \mapsto x - y^2 + z^2, \quad y \mapsto y, \quad z \mapsto z, \quad t \mapsto t,$$

we obtain

$$g = 2dxdt + 2x^2(dt)^2 + \frac{(dy)^2}{2y^2} + \frac{(dz)^2}{2y^2}.$$

This metric is decomposible and its holonomy algebra coincides with $\mathfrak{so}(1,1) \oplus \mathfrak{so}(2)$, but not with $\mathfrak{sim}(2)$.

Thus we get metrics with the holonomy algebra $\mathfrak{sim}(2)$ for the first time (even more, we find all such metrics in all dimensions).

Sketch of the proof of the Main Theorem.

$$g = 2dvdu + h + 2Adu + H(du)^2,$$

where

$h = h_{ij}(x^1, ..., x^n, u)dx^i dx^j$ is an u-family of Riemannian metrics,
$A = A_i(x^1, \dots, x^n, u)dx^i$ is an u-family of one-forms,
$H = H(v, x^1, ..., x^n, u)$ is a local function on M.

We must solve The equation $W = 0$.

Consider the local frame

$$p = \partial_v, \quad X_i = \partial_i - A_i\partial_v, \quad q = \partial_u - \frac{1}{2}H\partial_v.$$

Let E be the distribution generated by the vector fields X_1,...,X_n. Clearly, the vector fields p, q are isotropic, $g(p,q) = 1$, the restriction of g to E is positive definite, and E is orthogonal to p and q. The vector field p defines the parallel distribution of isotropic lines and it is recurrent, i.e. $\nabla p = \theta \otimes p$.

Curvature of the walker metric

$$\begin{aligned} R(p,q) &= -\lambda p \wedge q - p \wedge \vec{v}, \\ R(X,Y) &= R_0(X,Y) - p \wedge (P(Y)X - P(X)Y), \\ R(X,q) &= -g(\vec{v},X)p \wedge q + P(X) - p \wedge T(X), \qquad R(p,X) = 0 \end{aligned}$$

for all $X, Y \in \Gamma(E)$.

λ is a function,
$\vec{v} \in \Gamma(E)$,
$T \in \Gamma(\mathrm{End}(E))$ is symmetric, $T^* = T$,
$R_0 = R(h)$,
$P \in \Gamma(E^* \otimes \mathfrak{so}(E))$

$$\lambda = \frac{1}{2}\partial_v^2 H, \quad \vec{v} = \frac{1}{2}\left(\partial_i\partial_v H - A_i\partial_v^2 H\right) h^{ij} X_j,$$

$$h_{il}P^l_{jk} = -\frac{1}{2}\nabla_k F_{ij} + \frac{1}{2}\nabla_k \dot{h}_{ij} - \dot{\Gamma}^l_{kj} h_{li},$$

$$\begin{aligned} T_{ij} =& \frac{1}{2}\nabla_i\nabla_j H - \frac{1}{4}(F_{ik} + \dot{h}_{ik})(F_{jl} + \dot{h}_{jl})h^{kl} - \frac{1}{4}(\partial_v H)(\nabla_i A_j + \nabla_j A_i) \\ & - \frac{1}{2}(A_i\partial_j\partial_v H + A_j\partial_i\partial_v H) - \frac{1}{2}(\nabla_i \dot{A}_j + \nabla_j \dot{A}_i) \\ & + \frac{1}{2}A_iA_j\partial_v^2 H + \frac{1}{2}\ddot{h}_{ij} + \frac{1}{4}\dot{h}_{ij}\partial_v^2 H, \end{aligned}$$

where

$$F = dA, \quad F_{ij} = \partial_i A_j - \partial_j A_i$$

The Ricci operator

$$\begin{aligned}\mathrm{Ric}(p) =&\lambda p, \quad \mathrm{Ric}(X) = -g(X, \widetilde{\mathrm{Ric}}\, P - \vec{v})p + \mathrm{Ric}(h)(X),\\ \mathrm{Ric}(q) =& -(\mathrm{tr}\, T)p - \widetilde{\mathrm{Ric}}(P) + \vec{v} + \lambda q,\end{aligned}$$

where $\widetilde{\mathrm{Ric}}\, P = h^{ij} P(X_i) X_j$

Scalar curvature: $s = 2\lambda + s_0$

Weyl tensor

$$W = R + R_L,$$

where the tensor R_L is defined by

$$R_L(X,Y) = LX \wedge Y + X \wedge LY,$$

$$L = \frac{1}{d-2}\left(\mathrm{Ric} - \frac{s}{2(d-1)}\,\mathrm{Id}\right)$$

$d = n+2$ is the dimension

$$R_L(p,X)=\frac{1}{n}p\wedge\left(\mathrm{Ric}(h)+\frac{(n-1)\lambda-s_0}{n+1}\,\mathrm{id}\right)X,$$

$$R_L(p,q)=\frac{1}{n}\left(\frac{2n\lambda-s_0}{n+1}p\wedge q+p\wedge(\vec{v}-\widetilde{\mathrm{Ric}}\,P)\right),$$

$$R_L(X,Y)=\frac{1}{n}\Big(p\wedge(g(X,\vec{v}-\widetilde{\mathrm{Ric}}\,P)Y-g(Y,\vec{v}-\widetilde{\mathrm{Ric}}\,P)X)$$
$$+\left(\mathrm{Ric}(h)-\frac{s}{2(n+1)}\right)X\wedge Y+X\wedge\left(\mathrm{Ric}(h)-\frac{s}{2(n+1)}\right)Y$$

$$R_L(X,q)=\frac{1}{n}\Big((\mathrm{tr}\,T)p\wedge X+g(X,\vec{v}-\widetilde{\mathrm{Ric}}\,P)p\wedge q+X\wedge(\vec{v}-\widetilde{\mathrm{Ric}}\,P)$$
$$+\left(\mathrm{Ric}(h)+\frac{(n-1)\lambda-s_0}{n+1}\,\mathrm{id}\right)X\wedge q\Big).$$

Lemma The equation $W=0$ is equivalent to the following system of equations:

$$s_0=-n(n-1)\lambda,\quad R_0=-\frac{1}{2}\lambda R_{\mathrm{id}},\quad P(X)=\vec{v}\wedge X,\quad T=f\,\mathrm{id}_E,$$

where X is any section of E and f is a function. In particular, $W=0$ implies that $\widetilde{\mathrm{Ric}}\,P=-(n-1)\vec{v}$ and the Weyl tensor W_0 of h is zero.

From the lemma it follows that

$\partial_v \lambda = 0$, hence

$$H = \lambda v^2 + H_1 v + H_0, \quad \partial_v H_1 = \partial_v H_0 = 0.$$

Each metric in the family $h(u)$ is of constant sectional curvature with the scalar curvature $s_0 = -n(n-1)\lambda$.

The coordinates can be chosen in such a way that

$$h = \Psi \sum_{k=1}^{n} (dx^k)^2, \quad \Psi = \frac{4}{\left(1 - \lambda(u) \sum_{k=1}^{n} (x^k)^2\right)^2}.$$

Now we must find the 1-form A and the functions H_1 and H_0.

Part of the equations:

$$f_1\delta_{ij} = \frac{1}{2}\nabla_i\nabla_j H_1 - \lambda(u)\frac{1}{2}(\nabla_i A_j + \nabla_j A_i).$$

These equations are equivalent to

$$\nabla_i Z_i = \nabla_j Z_j, \quad \nabla_i Z_j + \nabla_j Z_i = 0, \qquad i \neq j,$$

where

$$Z_i = \lambda A_i - \frac{1}{2}\partial_i H_1$$

and to

$$\partial_i \left(\frac{Z_i}{\Psi}\right) = \partial_j \left(\frac{Z_j}{\Psi}\right), \quad \partial_i \left(\frac{Z_j}{\Psi}\right) + \partial_j \left(\frac{Z_i}{\Psi}\right) = 0, \quad i \neq j.$$

Let $n \geq 3$. Then

$$Z_i = \Psi \left(x^i B_k(u) x^k - \frac{1}{2} B_i(u) \sum_{k=1}^{n} (x^k)^2 + d_{ik}(u) x^k + c(u) x^i + c_i(u) \right).$$

Next,

$$\lambda F_{ij} = \partial_i Z_j - \partial_j Z_i = \Psi^{\frac{3}{2}} \left((B_i - 2\lambda C_i) x^j - (B_j - 2\lambda C_j) x^i + 2\lambda x^k (d_{jk} x^i - d_{ik} x^j) \right)$$

This implies that $B_i(u) = \lambda(u) \tilde{B}_i(u)$ for some functions $\tilde{B}_i(u)$.

Another equation: if i, j, k are pair-wise different, then

$$0=\nabla_k F_{ij}=\partial_k F_{ij}-\frac{1}{\Psi}F_{ij}\partial_k\Psi-\frac{1}{2\Psi}F_{kj}\partial_i\Psi-\frac{1}{2\Psi}F_{ik}\partial_j\Psi$$
$$=\Psi\partial_k\left(\frac{F_{ij}}{\Psi}\right)-\lambda F_{kj}x^i\sqrt{\Psi}-\lambda F_{ik}x^j\sqrt{\Psi}.$$

Consequently,

$$-\partial_k\left(\frac{F_{ij}}{\Psi}\right)=\lambda x^k\Psi\big((\tilde{B}_j-2C_j)x^i-(\tilde{B}_i-2C_i)x^j\big)+2\sqrt{\Psi}(x^i d_{kj}-x^j d_{ki})\big)+2\lambda$$

Integrating over x^k, we get

$$F_{ij}=\Psi^{\frac{3}{2}}\big((\tilde{B}_i-2C_i)x^j-(\tilde{B}_j-2C_j)x^i+2(d_{li}x^j-d_{lj}x^i)x^l\big)-\Psi C_{ij}(x^i,x^j,u)$$

Using other equations, we get

$$F_{ij} = \Psi^{\frac{3}{2}}\left(4C_j(u)x^i - 4C_i(u)x^j + \lambda(u)(C_{li}(u)x^j - C_{lj}(u)x^i)x^l\right) - \Psi C_{ij}(u).$$

Recall that $F = dA$.

the transformation $v \mapsto v - \phi(x^1, \dots, x^m, u)$ changes A_i to $A_i + \partial_i \phi$ and does not change F. This allows us to choose any A such that $dA = F$. We take

$$A_i = \Psi\left(-4C_k(u)x^k x^i + 2C_i(u)\sum_{k=1}^{n}(x^k)^2 + \frac{1}{2}C_{ik}(u)x^k\right).$$

We are left with the equations

$$\partial_i H_1 = -2\Psi \left(2\lambda C_k(u) x^k x^i - \lambda C_i(u) \sum_{k=1}^{n} (x^k)^2 + \frac{1}{2}\dot{\lambda} x^i + C_i(u) \right)$$

$$\partial_i \partial_j \frac{H_0}{\sqrt{\Psi}} = 2\Psi^{\frac{3}{2}} x^i x^j \sum_{k=1}^{n} C_k^2(u).$$

$$H_1 = -4C_k(u) x^k \sqrt{\Psi} - \partial_u \ln \Psi + K(u)$$

$$H_0 = \frac{4}{\lambda^2(u)} \Psi \sum_{k=1}^{n} C_k^2(u) + \sqrt{\Psi} \sum_{k=1}^{n} f_k(x^k, u),$$

$$f_i(x^i, u) = a(u)(x^i)^2 + D_i(u) x_i + d_i(u)$$

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where

$$\Psi=\frac{4}{(1-\lambda(u)\sum_{k=1}^n(x^k)^2)^2},$$

$$A=A_idx^i,\quad A_i=\Psi\left(-4C_k(u)x^kx^i+2C_i(u)\sum_{k=1}^{n}(x^k)^2\right),$$

$$H_1=-4C_k(u)x^k\sqrt{\Psi}-\partial_u\ln\Psi+K(u),$$

$$s=-(n-2)(n+1)\lambda(u)$$