FUNDAMENTAL INVARIANTS OF SYSTEMS OF ODES OF HIGHER ORDER

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Abstract. We find the complete set of fundamental invariants for systems of ordinary differential equations of order ≥ 4 under the group of point transformations generalizing similar results for contact invariants of a single ODE and point invariants of systems of the second and the third order.

It appears that starting from systems of the k-th order, $k \geq 4$, the complete set of fundamental invariants is formed by $(k-1)$ generalized Wilczynski invariants coming from the linearized system and the one additional invariant of degree 2.

1. Introduction

1.1. **Notion of invariants of differential equation.** The geometry and local equivalence of ordinary differential equations has a long history. It starts with pioneer works of Sophus Lie who introduced the notion of Lie pseudogroups and suggested a method of finding invariants of various geometric objects under the action of a certain pseudogroup.

In this paper we consider systems of ordinary differential equations viewed up to so-called pseudogroup of point transformations. In coordinate notation this means we treat systems of $m \geq 2$ equations of order $k \geq 2$:

$$(1) \quad y^i(x)^{(k)} = F^i(x, y^j(x)^{(l)}),\ 1 \leq i \leq m;\ 1 \leq j \leq m, 0 \leq l \leq k-1.$$

up to arbitrary locally invertible changes of independent variable x and dependent variables y^i, $1 \leq i \leq m$.

By a *(point) invariant of order r* of system (1) we mean a function depending on the right hand side of (1) and its partial derivatives up to order r that is not changed under the action of the pseudogroup of point transformations. In the current paper we shall outline a method to construct (all) point invariants of (1), but will but actually compute them explicitly. Instead we shall work with so-called *relative invariants*

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and find the minimal set of them that generates all other invariants (absolute and relative) via operation of invariant differentiation. We call such set of relative invariants *fundamental invariants*.

Naturally, all non-constant invariants constructed in this paper will vanish for the *trivial system of equations*, i.e., the system of form (1) with vanishing right hand side. As an immediate application, we get an explicit characterization of the class of *trivializable systems of ODEs*, i.e., the systems that are equivalent to the trivial equation under point transformations. So, fundamental system of invariants can be thought as a minimal set of differential relations describing the orbit of the trivial equation.

The notions of relative and absolute invariants can be formalized in a rigorous way using the language of jet spaces. Namely, first we consider the jets $J^k(\mathbb{R},\mathbb{R}^m)$ of maps from $\mathbb{R}$ to $\mathbb{R}^m$ along with natural projections $\pi_{k,l}\colon J^k(\mathbb{R},\mathbb{R}^m)\to J^l(\mathbb{R},\mathbb{R}^m)$ for any $k\ge l\ge 0$. Then the k-th order system of ODEs is a submanifold $\mathcal{E}$ of codimension k in $J^k(\mathbb{R},\mathbb{R}^m)$ transversal to fibers of the projection $\pi_{k,k-1}$. Locally this is equivalent to defining the section $s\colon J^{k-1}(\mathbb{R},\mathbb{R}^m)\to J^k(\mathbb{R},\mathbb{R}^m)$ of the projection $\pi_{k,k-1}$, and equations (1) are just coordinate expressions of this section. A solution of the equation E is an arbitrary map $y\colon \mathbb{R}\supset U\to\mathbb{R}^m$ such that its k-th prolongation $y^{(k)}=\{[y]^k_x \mid x\in U\}$ belongs to $\mathcal{E}$. Here by $[y]^k_x$ we denote the k-jet of map y at $x\in\mathbb{R}$.

Recall that we can identify $J^0(\mathbb{R},\mathbb{R}^m)$ with $\mathbb{R}\times\mathbb{R}^m=\mathbb{R}^{m+1}$, and any local diffeomorphism $\phi\colon\mathbb{R}^{m+1}\to\mathbb{R}^{m+1}$ has a unique prolongation $\phi^{(k)}\colon J^k(\mathbb{R},\mathbb{R}^m)\to J^k(\mathbb{R},\mathbb{R}^m)$ compatible with prolongation operation as well as with projections $\pi_{k,l}$. The family of all prolongations $\phi^{(k)}$ is exactly the Lie pseudogroup of point transformations $\mathcal{D}^k(\mathbb{R},\mathbb{R}^m)$ acting on submanifolds $\mathcal{E}\subset J^k(\mathbb{R},\mathbb{R}^m)$ corresponding to the k-th order systems of ODEs.

Denote by $\mathbf{J}^{k,r}$ the space of r-jets of all equations submanifolds E in $J^k(\mathbb{R},\mathbb{R}^m)$. It is an open subset in the manifold of r-jets of all submanifolds of codimension k in $J^k(\mathbb{R},\mathbb{R}^m)$. The action of the pseudogroup $\mathcal{D}^k=\mathcal{D}^k(\mathbb{R},\mathbb{R}^m)$ is naturally prolonged to $\mathbf{J}^{k,r}$.

Definition 1. An *(absolute) invariant of order r for the k-th order systems of ODEs* is a function on $J^{k,r}$ (or on its open dense $\mathcal{D}^k$-invariant subset) which is preserved by the prolonged action of $\mathcal{D}^k$. A *relative invariant of order* r is a $\mathcal{D}^k$ equivariant map $I\colon \mathbf{J}^{k,r}\to V$ taking values in a finite-dimensional representation of the pseudogroup $\mathcal{D}^k$.

Let us list a number of known invariants for a single ODE and systems of ODEs in low orders.

Example 1. One of the simplest examples is a class of second order ODEs $y'' = F(x, y, y')$ viewed up to the pseudogroup of all point transformations. The complete set of invariants in this case was computed by Tresse [25] at the end of the 19-th century. It appears that the simplest non-trivial relative invariants appear only in the order 4:

$$I_1 = F_{1111} = \frac{\partial^4 F}{(\partial y')^4};$$
$$I_2 = \frac{1}{6}F_{11xx} - \frac{1}{6}F_1 F_{11x} - \frac{2}{3}F_{01x} + \frac{2}{3}F_1 F_{01} + F_{00} - \frac{1}{2}F_0 F_{11}.$$

Here we use the standard classical notation used in case of a single ODE of order k: the subscript $i = 0, \dots, k-1$ means the partial derivative with respect to $y^{(i)}$ and the subscript x means the total derivative:

$$\frac{d}{dx} = \frac{\partial}{\partial x} + y^{(1)}\frac{\partial}{\partial y} + \cdots + y^{(k-1)}\frac{\partial}{\partial y^{(k-2)}} + F\frac{\partial}{\partial y^{(k-1)}}.$$

The geometry lying behind this equation was explored by E. Cartan [2], who associated the the canonical coframe with this equation and proved that all its invariants can be derived by covariant differentiation from the above two relative invariants. In particular, the general equation is equivalent to the trivial one if and only if $I_1 = I_2 = 0$. See also [16] for the interpretation of these invariants in terms of the associated Fefferman metric.

Example 2. In the case of a singe ODE of the order ≥ 3 there are two kinds of pseudogroups (and invariants) typically considered in the applications. Namely, the largest pseudogroup acting on ODEs of the fixed order (and preserving this order) is a so-called *pseudogroup of contact transformations*. These are the transformations of $J^1(\mathbb{R}, \mathbb{R})$ the form $\phi^{(k-1)}$, where ϕ is a local transformation of $J^1(\mathbb{R}, \mathbb{R})$ preserving the natural contact structure on it.

The equivalence problem of the 3rd order ODEs up to contact transformations was studied by S.-S. Chern [4], who found the following set of fundamental invariants in this case:

$$I_1 = F_{2222} = \frac{\partial^4 F}{(\partial y'')^4};$$
$$W = -F_0 - \frac{1}{3}F_1 F_2 - \frac{2}{27}F_2^3 + \frac{1}{2}F_{1x} + \frac{1}{3}F_2 F_{2x} - \frac{1}{6}F_{2xx}.$$

The second of these invariants was first found by Wünschmann [27] back in 1905 and is usually called *Wünschmann invariant* in his honor.

The point geometry of the 3rd order ODEs is quite different and was studied by Elie Cartan [3]. It appears that a given equation $y''' =$

$F(x, y, y', y'')$ is equivalent to the trivial one up to point transformations if and only if the following four invariants vanish identically:

$$W = F_{222} = F_{22}^2 + 6F_{122} + 2F_2F_{222} = 0;$$
$$F_{11} + 2W_2 - 2F_{02} + \tfrac{2}{3}F_2F_{12} + 2F_{22}\left(\tfrac{1}{3}F_{qx} - \tfrac{2}{9}F_2^2 - F_1\right) = 0,$$

where W is the above Wünschmann invariant.

Example 3. Systems of the second order were studies in the work of Mark Fels [9]. Starting from an arbitrary system of the 2nd order ODEs:

$$(y_i)'' = F_i(x, y_j, y_k'), \quad i = 1, \dots, m,$$

he constructs an absolute parallelism solving the equivalence problem as well as two fundamental invariants. They appear in the degree 2 and 3 and are equal to trace free parts of the following expressions:

$$(W_2)^i_j = \operatorname{tr}_0\left(\frac{\partial F_i}{\partial y_j} - \frac{1}{2}\frac{d}{dx}\left(\frac{\partial F_i}{\partial p_j}\right) + \frac{1}{4}\sum_{r=1}^{m}\frac{\partial F_i}{\partial p_r}\frac{\partial F_r}{\partial p_j}\right);$$
$$(I_3)^i_{jkl} = \operatorname{tr}_0\left(\frac{\partial^3 F_i}{\partial p_j \partial p_k \partial p_l}\right),$$

where by p_i we denote y_i' , $i = 1, \dots, m$, which together with x and y_j form local coordinates on $J^1(\mathbb{R}, \mathbb{R}^m)$, and, as above, $\frac{d}{dx}$ denotes the operator of total derivative with respect to x.

We note that in case of systems of ordinary differential equations the classes of point and contact transformations coincide and we shall always consider the pseudogroup of point transformations in case of systems of ODEs without mentioning this explicitly.

1.2. **Generalized Wilczynski invariants.** Wilczynski invariants were first introduced by Wilczynski [26] as a fundamental set of invariants for linear scalar ODEs on one function $y(x)$ viewed up to Lie preudogroup $(x, y) \mapsto (\lambda(s), \mu(x)y)$. The generalization of Wilczynski invariants to the systems of linear ODEs was obtained by Se-ashi [20].

Consider an arbitrary system of linear ordinary differential equations of order $k + 1$:

$$y^{(k+1)} + P_k(x)y^{(k)} + \cdots + P_0(x)y(x) = 0,$$

where $y(x)$ is an $\mathbb{R}^m$-valued vector function. The canonical Laguerre–Forsyth form of these equations is defined by conditions $P_k = 0$ and

$\operatorname{tr} P_{k-1} = 0$. Then the following expressions:

(2)
$$\Theta_r = \sum_{j=1}^{r-1} (-1)^{j+1} \frac{(2r-j-1)!(k-r+j)!}{(r-j)!(j-1)!} P_{k-r+j}^{(j-1)}, \quad r = 2, \dots, k+1.$$

are the $\operatorname{End}(\mathbb{R}^m)$-valued relative invariants, where each invariant Θ_r has a degree r. Note the first non-trivial generalized Wilczynski invariant has degree 2 and is trace-free. In particular, it vanishes identically in the scalar case, but is a non-trivial invariant in case of systems of $m \geq 2$ linear equations.

Generalized Wilczysnki invariants W_r, $r = 2, \dots, k+1$ of system (1) are defined as invariants Θ_r evaluated at the linearization of the system. Formally they are obtained from (2) by substituting $P_r(x)$ by matrices $-\left(\frac{\partial f^i}{\partial (y^j)^{(r)}}\right)$ and the usual derivative d/dx by the total derivative. See [7] for more details.

For example, the Wünschmann invariant W for scalar 3rd order ODE is exactly the lowest degree generalized Wilczynski invariant in case of scalar ODEs, and the above invariant W_2 for systems of 2nd order is the simplest non-trivial example of a generalized Wilczynski invariant in case of systems of ODEs.

Example 4. Fundamental invariants for systems of the third order were computed in the works of Alexandr Medvedev [12]. Denote trace free part of the tensor as tr_0. Then for the systems of m ODEs of the 3rd order the following tensors form the minimal set of fundamental invariants:

(1) two generalized Wilczynski invariants W_2 and W_3:

$$(W_2)^i_j = \operatorname{tr}_0 \left(\frac{\partial f^i}{\partial p^j} - \frac{d}{dx} \frac{\partial f^i}{\partial q^j} + \frac{1}{3} \frac{\partial f^i}{\partial q^k} \frac{\partial f^k}{\partial q^j} \right),$$

$$\begin{aligned}(W_3)^i_j = & \frac{\partial f^i}{\partial y^j} + \frac{1}{3} \frac{\partial f^i}{\partial q^k} \frac{\partial f^k}{\partial p^j} - \frac{d}{dx} \frac{\partial f^i}{\partial p^j} + \frac{2}{3} \frac{d^2}{dx^2} \frac{\partial f^i}{\partial q^j} + \frac{2}{27} \left(\frac{\partial f^i}{\partial q^j} \right)^3 \\ & - \frac{4}{9} \frac{\partial f^i}{\partial q^k} \frac{d}{dx} \frac{\partial f^k}{\partial q^j} - \frac{2}{9} \frac{d}{dx} \left(\frac{\partial f^i}{\partial q^k} \right) \frac{\partial f^k}{\partial q^j} - 2\delta^i_j \frac{d}{dx} H^x;\end{aligned}$$

(2) two additional invariants of the degree 2 and 4:

$$(I_2)^i_{jk} = \operatorname{tr}_0 \left(\frac{\partial^2 f^i}{\partial q^j \partial q^k} \right),$$

$$(I_4)_{jk} = - \frac{\partial H^{-1}_k}{\partial p_j} + \frac{\partial}{\partial q_j} \frac{\partial}{\partial q_k} H^x - \frac{\partial}{\partial q_k} \frac{d}{dx} H^{-1}_j$$

$$- \frac{\partial}{\partial q^k}\left(H_l^{-1}\frac{\partial f^l}{\partial q^j}\right) + 2H_j^{-1}H_k^{-1},$$

where

$$H_j^{-1} = \frac{1}{6(m+1)}\left(\frac{\partial^2 f^i}{\partial q^i \partial q^j}\right),$$

$$H^x = -\frac{1}{4m}\left(\frac{\partial f^i}{\partial p^i} - \frac{d}{dx}\frac{\partial f^i}{\partial q^i} + \frac{1}{3}\frac{\partial f^i}{\partial q^k}\frac{\partial f^k}{\partial q^i}\right).$$

Example 5. Fundamental invariants for scalar ODEs of order 4 where computed by Robert Bryant [1] and for orders ≥ 5 by Boris Doubrov [6]. It appears that apart from generalized Wilczynski invariants there are only 2 additional invariants for ODEs of degree 4 and 6 and three additional invariants for orders 5 and ≥ 7. They are given by:

- invariant $I_3 = F_{333}$ for 4th order ODE and invariant $I_2 = F_{k,k}$ for $(k+1)$-th order ODE, $k+1 \geq 5$;
- extra invariants:
  $k+1=4$: $J_4 = F_{233} + \frac{1}{6}F_{33}^2 + \frac{9}{8}F_3F_{333} + \frac{3}{4}F_{333x}$;
  $k+1=5$: $J_5 = F_{234} - \frac{2}{3}F_{333} - \frac{1}{2}F_{34}^2 \mod I_2, W_3$;
  $k+1 \geq 6$: $J_3 = F_{k,k-1} \mod I_2$;
  $k=1 \geq 7$: $J_4 = F_{k-1,k-1} \mod I_2, J_3, W_3$.

The main result of this paper is the computation of fundamental invariants in all remaining cases, namely for systems of ODEs of order ≥ 4.

Theorem. *The following relative invariants form a fundamental set for systems of $m \geq 2$ ODEs of the order $k+1 \geq 4$:*

(1) *generalized Wilczynski invariants W_r of the degree r, $2 \leq r \leq k+1$;*
(2) *one additional invariant I_2 of the degree 2:*

$$(I_2)^i_{jl} = \operatorname{tr}_0\left(\frac{\partial^2 f^i}{\partial y^j_k \partial y^l_k}\right),$$

where tr_0 *denotes the trace-free part of the tensor.*

The paper is organized as follows. In Section 2 we recall the construction of the canonical Cartan connection from [5] and show that the fundamental set of invariants is described by the cohomology space $H^2_+(\mathfrak{g}_-, \mathfrak{g})$, where $\mathfrak{g}$ is a symmetry algebra of the trivial system of ODEs equipped with an appropriate grading. It is defined in Subsection 2.2.

We compute this cohomology space in Section 3 using the Serre–Hochschild spectral sequence. Part of this cohomology corresponds to the invariants of the underlying non-holonomic distribution. In our case this non-holonomic distribution is a contact distribution on the jet space $J^k(\mathbb{R}, \mathbb{R}^m)$. It is a flat distribution of type $\mathfrak{g}_-$, where $\mathfrak{g}_-$ can also be viewed as the Tanaka symbol of the contact distribution.

Let $\bar{\mathfrak{g}}$ be the Tanaka prolongation of $\mathfrak{g}_-$, which is infinite-dimensional and contains $\mathfrak{g}$ as a finite-dimensional subalgebra. We consider the natural map of cohomology spaces $\gamma\colon H^2(\mathfrak{g}_-, \mathfrak{g}) \to H^2(\mathfrak{g}_-, \bar{\mathfrak{g}})$ and show that only the kernel of this map on $H^2_+(\mathfrak{g}_-, \mathfrak{g})$ corresponds to non-vanishing invariants. The kernel of this map is computed in Subsection 3.2. In particular, we show that for systems of ODEs of order ≥ 4 there is exactly one fundamental invariant in degree 2 in addition to generalized Wilczysnki invariants.

Finally, in Section 4 we explicitly compute the coefficients of the canonical Cartan connection in degrees 1 and 2 and derive the explicit formula of this additional invariant of degree 2.

2. Canonical Cartan connection

One of the main techniques for computing the fundamental invariants for differential equations is Cartan's equivalence method and its further generalization of N. Tanaka in the context of so-called nilpotent differential geometry. The main advantage of Tanaka's approach is the adaptation of all constructions to the underlying non-holonomic vector distribution (the contact distribution on the jet spaces), the powerful algebraic techniques for constructing a canonical Cartan connection (instead of much weaker absolute parallelism structures) and the way to describe the principal part of the curvature (in our terminology this is exactly the set of fundamental invariants) via cohomology of finite-dimensional graded Lie algebras. In this section we outline how the geometry of systems of ODEs fits into the framework of nilpotent differential geometry. Further details and references can be found in [5].

2.1. **System of ODEs as a filtered manifold.** Consider a manifold $M = \mathbb{R}^{m+1}$ with coordinates (x, y^i), where $i = 1, \dots, m$. An arbitrary system of m ordinary differential equations of the order $k+1$ has the form:

$$(3) \qquad \left(y^i\right)^{(k+1)}(x) = f^i\left(\left(y^j\right)^{(s)}(x), x\right), \quad s = 0, \dots, k,$$

where $i, j = 1, \dots, m$, $s = 0, \dots, k$, $m \geq 2$ and functions f^i are smooth.

Let $J^{k+1} = J^{(k+1)}(\mathbb{R}, \mathbb{R}^m)$ be the space of $(k+1)$-jets of smooth maps from $\mathbb{R}$ to $\mathbb{R}^m$. We use the standard local coordinate system in

$J^{(k+1)}(\mathbb{R},\mathbb{R}^m)$:

$$(x, y^i_a), \quad 0 \le a \le k+1, 1 \le i \le m,$$

where we assume that y^i_a has the meaning of a-th derivative of $y^i(x)$.

Equations (3) define a submanifold $\mathcal{E}$ in $J^{k+1}(\mathbb{R},\mathbb{R}^m)$ by:

$$y^i_{k+1} = f^i\left(y^j_a, x\right), \quad i = 1,\dots,m, a = 0,\dots,k.$$

The projection $\pi_{k+1,k}\colon J^{k+1} \to J^k$ establishes a (local) diffeomorphism $\mathcal{E}$ with $J^{(k)}$. With every system of ODEs we associate a pair of distributions on $\mathcal{E}$:

$$\begin{aligned} E &= \left\langle \frac{\partial}{\partial x} + \sum_{a=1}^{k} y^i_a \frac{\partial}{\partial y^i_{a-1}} + f^i \frac{\partial}{\partial y^i_k} \right\rangle, \\ V &= \left\langle \frac{\partial}{\partial y^i_k} \right\rangle, \end{aligned}$$

where $i = 1,\dots,m$. A distribution $C^{-1} = E \oplus V$ is mapped to the standard contact distribution on J^k under local diffeomorphism $\pi_{k+1,k}\colon \mathcal{E} \to J^k$. In particular, it is bracket generating, and its weak derived series defines a filtration of the tangent bundle $T\mathcal{E}$:

$$C^{-1} \subset C^{-2} \subset \dots \subset C^{-k-1} = T\mathcal{E},$$

where $C^{-i-1} = C^{-i} + [C^{-i}, C^{-1}]$.

Define the following coframe on $\mathcal{E}$:

$$\begin{aligned} \theta_x &= dx, \\ \theta^i_{-1} &= dy^i_k - f^i\,dx, \\ \theta^i_{-r} &= dy^i_{k+1-r} - y^i_{k+2-r}\,dx, \quad 2 \le r \le k+1. \end{aligned}$$

We call an arbitrary coframe ω_x, ω^i_{-r}, $1 \le i \le m, 1 \le r \le k+1$, on $\mathcal{E}$ *adapted to the equation* (3), if

(a) the annihilator of forms $\omega^i_{-r-1},\dots,\omega^i_{-k-1}$ is equal to C^{-r} for all $r = 1,\dots,k$;
(b) the annihilator of forms $\omega^i_{-1},\dots,\omega^i_{-k-1}$ is equal to E;
(c) the annihilator of $\omega_x, \omega^i_{-2},\dots,\omega^i_{-k-1}$ is equal to F.

We call an adapted coframe *regular*, if in addition it satisfies the condition:

(d) $d\omega^i_{-r} + \omega_x \wedge \omega^i_{-r+1} = 0 \mod \langle \omega^j_{-r},\dots,\omega^j_{-1}\rangle$ for all $2 \le r \le k+1$.

2.2. **Adapted Cartan connections.** Let $\mathfrak{g}$ be the symmetry algebra of the trivial system of m ODEs of order $(k+1)$. In the sequel we assume that $k \geq 3$, $m \geq 2$, that is we consider only systems of ordinary differential equations of an order ≥ 4. Under these conditions the Lie algebra $\mathfrak{g}$ is isomorphic to the semidirect product of a reductive Lie algebra $\mathfrak{a} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{gl}(m,\mathbb{R})$ and an abelian ideal $V = V_k \otimes W$, where V_k is an irreducible $\mathfrak{sl}(2,\mathbb{R})$-module isomorphic to $S^k(\mathbb{R}^2)$ and $W = \mathbb{R}^m$ is the standard $\mathfrak{gl}(m,\mathbb{R})$-module.

Let us fix a basis of $\mathfrak{g}$. Let:

$$x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

be a basis for $\mathfrak{sl}(2,\mathbb{R})$. Fix a basis of $\mathfrak{sl}(2,\mathbb{R})$-module V_k consisting of elements $v^i = f_2^{k-i} f_1^i / i!$, where f_1, f_2 is the standard basis in $\mathbb{R}^2$. We denote by $\{e_1, \dots, e_m\}$ and $\{e_j^i\}$ the standard bases of $\mathbb{R}^m$ and $\mathfrak{gl}(m,\mathbb{R})$ respectively. (That is we have $e_j^i e_k = \delta_{ik} e_j$.)

The degrees of elements in the Lie algebra $\mathfrak{g}$ are defined as follows:

$$\begin{aligned} \mathfrak{g}_1 &= \mathbb{R}y, \\ \mathfrak{g}_0 &= \mathbb{R}h \oplus \mathfrak{gl}(m,\mathbb{R}), \\ \mathfrak{g}_{-1} &= \mathbb{R}x \oplus \mathbb{R}v^k \otimes W, \\ \mathfrak{g}_{-i} &= \mathbb{R}v^{k+1-i} \otimes W, \quad i = 2, \dots, k+1, \end{aligned}$$

and $\mathfrak{g}_n = \{0\}$ for all other $n \in \mathbb{Z}$. As we see, the negative part $\mathfrak{g}_-$ of $\mathfrak{g}$ is equal to $\mathbb{R}x \oplus V$. We denote the non-negative part of the Lie algebra $\mathfrak{g}$ by $\mathfrak{h}$.

Globally, we can define a Lie group G as a semidirect product of $SL(2,\mathbb{R}) \times GL(m,\mathbb{R})$ and the commutative group $V = V_k \otimes \mathbb{R}^m$. Let H be the direct product of subgroup $ST(2,\mathbb{R})$ of all lower-triangular matrices in $SL(2,\mathbb{R})$ and $GL(m,\mathbb{R})$. Then G/H can be identified with the trivial equation $\mathcal{E}_0 \subset J^{k+1}(\mathbb{R},\mathbb{R}^m)$.

Our nearest goal is to construct a Cartan connection on $\mathcal{E}$, modeled by the homogeneous space G/H, that will be naturally associated with the equation (3). Such Cartan connection consists of a principal H-bundle $\pi\colon \mathcal{G} \to \mathcal{E}$ and the $\mathfrak{g}$-valued differential form ω on $\mathcal{G}$ such that

(1) $\omega(X^*) = X$ for all fundamental vector fields X^* on $\mathcal{G}$,$X \in \mathfrak{h}$;
(2) $R_h^* \omega = \operatorname{Ad} h^{-1} \omega$ for all $h \in H$;
(3) ω defines an absolute parallelism on $\mathcal{G}$.

Any $\mathfrak{g}$-valued form ω can be written as

$$\omega = \sum_{r=1}^{k+1} \omega_{-r}^i v^{k+1-r} \otimes e_i + \omega_x x + \omega_h h + \omega_j^i e_i^j + \omega_y y,$$

We say that a Cartan connection ω on a principal H-bundle $\pi\colon \mathcal{G} \to \mathcal{E}$ is *adapted to the equation* (3), if for any local section s of π the set $\{s^*\omega_x, s^*\omega^i_{-r}\}$ is an adapted coframe on $\mathcal{E}$. The Cartan connection ω is said to be *regular*, if the above coframe is also regular. It is easy to see that this definition does not depend on the choice of the local section s.

Denote by $C^k(\mathfrak{g}_-, \mathfrak{g})$ the space of all k-cochains on $\mathfrak{g}_-$ with values in $\mathfrak{g}$. Any Cartan connection ω modeled by the homogeneous space G/H determines the curvature tensor $\Omega = d\omega + 1/2[\omega, \omega]$ on $\mathcal{G}$ and the curvature function $c\colon \mathcal{G} \to C^2(\mathfrak{g}_-, \mathfrak{g})$, where

$$c_p(u, v) = \Omega_p(\omega_p^{-1}(u), \omega_p^{-1}(v)) \quad \text{for all } u, v \in \mathfrak{g}_-,\ p \in \mathcal{G}.$$

This function satisfies the condition

$$c(ph) = h^{-1}.c(p) \quad \text{for all } h \in H, p \in \mathcal{G}, \tag{4}$$

where H acts on $C^2(\mathfrak{g}_-, \mathfrak{g})$ in the natural way.

Since $\mathfrak{g}_-$ and $\mathfrak{g}$ are graded, all spaces $C^k(\mathfrak{g}_-, \mathfrak{g})$ inherit the gradation

$$C^k(\mathfrak{g}_-, \mathfrak{g}) = \sum_r C^k_r(\mathfrak{g}_-, \mathfrak{g}),$$

where

$$C^k_r(\mathfrak{g}_-, \mathfrak{g}) = \{\alpha \in C^k(\mathfrak{g}_-, \mathfrak{g}) \mid \alpha(\mathfrak{g}_{i_1}, \dots, \mathfrak{g}_{i_k}) \subset \mathfrak{g}_{i_1+\dots+i_k+r}\}.$$

The standard cochain differential

$$\partial\colon C^k(\mathfrak{g}_-, \mathfrak{g}) \to C^{k+1}(\mathfrak{g}_-, \mathfrak{g})$$

preserves this gradation. Decompose the curvature function c to the sum $c = \sum_r c_r$, where each c_r takes values in $C^2_r(\mathfrak{g}_-, \mathfrak{g})$. It is easy to see that regularity of the Cartan connection implies that $c_r = 0$ for all $r \leq 0$. In other words, the curvature function of a regular Cartan connection takes values in $C^2_+(\mathfrak{g}, \mathfrak{g})$.

2.3. **Harmonic analysis on the symbol algebra.** Generally speaking, there are many regular Cartan connections adapted to a given equation (3). The basic idea of choosing a unique one among them is to add linear conditions on structure function. Finding these conditions is not easy since they should guarantee the existence and uniqueness of the required Cartan connection and at the same time they must be invariant with respect to the action of H on $C^2(\mathfrak{g}_-, \mathfrak{g})$ because of property (4) of the curvature function c. Hopefully, as it was shown by T. Morimoto [14, 15], they can be derived from the "harmonic theory" on our symbol Lie algebra $\mathfrak{g}$.

First, we fix a scalar product $(,)$ on $\mathfrak{g}$ such that vectors $x, y, h, v^i \otimes e_j, e^i_j$ form an orthogonal basis and

$$\langle e^i_j, e^i_j \rangle = 1, \quad (v^i \otimes e_j, v^i \otimes e_j) = (k-i)!/i!, \quad 0 \le i \le k;$$
$$(x,x) = (y,y) = 1, (h,h) = 2.$$

This metric $(,)$ is chosen in such a way that

(1) all spaces $\mathfrak{g}_i$ are mutually orthogonal;
(2) $(S,T) = \operatorname{tr} {}^tST$ for all $S, T \in \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{gl}(m,\mathbb{R})$;
(3) $(Su, v) = (u, {}^tSv)$ for all $S \in \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{gl}(m,\mathbb{R})$, $u, v \in V$, so that the transposition with respect to this metric on V agrees with standard matrix transpositions in $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{gl}(m,\mathbb{R})$.

Then we extend this metric to the spaces $C^k(\mathfrak{g}_-, \mathfrak{g})$ in the standard way and denote by

$$\partial^* \colon C^{k+1}(\mathfrak{g}_-, \mathfrak{g}) \to C^k(\mathfrak{g}_-, \mathfrak{g})$$

the operator adjoint to the cochain differential ∂.

Finally, we add one more condition on our Cartan connection adapted to equation (3).

Proposition 1 ([5, 14])**.** *Among all Cartan connections adapted to equation* (3) *there exists a unique (up to isomorphism) Cartan connection whose structure function is co-closed, i.e.,* $\partial^* c = 0$.

In the sequel we call this Cartan connection *a normal Cartan connection associated with equation* (3) and denote it by $\omega_{\mathcal{E}}$.

2.4. **Fundamental invariants.** Normal Cartan connections give also an algorithm of constructing invariants of ordinary differential equations. Let $\omega_{\mathcal{E}} \colon T\mathcal{G} \to \mathfrak{g}$ be the normal Cartan connection associated with the equation $\mathcal{E}$. Since $\omega_{\mathcal{E}}$ defines an absolute parallelism on $\mathcal{G}$, we see that the algebra of its invariants is generated by coefficients of its structure function and their covariant derivatives. Using the special properties of the canonical Cartan connection and in particular its deep relation with harmonic analysis on the symbol algebra $\mathfrak{g}$, we may reduce the number of generators of the algebra of invariants of $\omega_{\mathcal{E}}$.

Proposition 2 ([5])**.** *The algebra of invariants of the canonical Cartan connection $\omega_{\mathcal{E}}$ is generated by the coefficients of the harmonic part of the structure function c of $\omega_{\mathcal{E}}$ and its covariant derivatives. In particular, the curvature function c vanishes if and only if its harmonic part vanishes.*

As the harmonic part of the cochain complex $C(\mathfrak{g}_-, \mathfrak{g})$ is naturally isomorphic to the corresponding Lie algebra cohomology $H(\mathfrak{g}_-, \mathfrak{g})$, the number of fundamental invariants and its degree can be determined by computing the cohomology spaces $H^2_+(\mathfrak{g}_-, \mathfrak{g})$. This will be done in the next section.

3. Cohomology of systems of ODEs of higher order

Consider the cohomology spaces $H^k(\mathfrak{g}_-, \mathfrak{g})$. They can be naturally supplied with the grading:

$$H^k(\mathfrak{g}_-, \mathfrak{g}) = \bigoplus_{p\in\mathbb{Z}} H^k_p(\mathfrak{g}_-, \mathfrak{g}),$$

where

$$H^k_p(\mathfrak{g}_-, \mathfrak{g}) = \{[c] \in H^k(\mathfrak{g}_-, \mathfrak{g}) \mid c(\mathfrak{g}_{i_1}, \dots, \mathfrak{g}_{i_k}) \subset \mathfrak{g}_{i_1+\dots+i_k+n}\}.$$

As mentioned above, the fundamental invariants of a system of the $(k+1)$-th order ODEs are described by the positive part of the second cohomology space $H^2(\mathfrak{g}_-, \mathfrak{g})$. Below we compute this space by means of the Serre–Hochschild spectral sequence, determined by the subalgebra V of $\mathfrak{g}_-$.

Recall (see [10, 8]) that the Serre–Hochschild spectral sequence is one of the main technical tools for computing cohomology $H(\mathfrak{g}, M)$ of an arbitrary Lie algebra $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module M in case when the Lie algebra $\mathfrak{g}$ has a non-trivial ideal $\mathfrak{a}$. In this case the second term E_2 of this spectral sequence is equal to $E_2^{p,q} = H^p(\mathfrak{g}/\mathfrak{a}, H^q(\mathfrak{a}, V))$.

In our case, since V is an ideal of $\mathfrak{g}_-$, the second term E_2 of the Serre-Hochschild spectral sequence computing $H(\mathfrak{g}_-, \mathfrak{g})$ has the form: $E_2 = \bigoplus_{p,q} E_2^{p,q}$, where

$$E_2^{p,q} = H^p(\mathbb{R}x, H^q(V, \mathfrak{g})), \quad p, q \geq 0.$$

We immediately get the following result regarding the structure of $H^2(\mathfrak{g}_-, \mathfrak{g})$:

Proposition 3. *The second cohomology space $H^2(\mathfrak{g}_-, \mathfrak{g})$ is naturally isomorphic with the subspace $E_2^{1,1} \oplus E_2^{0,2}$ of the Serre-Hochschild spectral sequence determined by the ideal $V \subset \mathfrak{g}_-$.*

Moreover, we have

$$E_2^{1,1} = H^1(\mathbb{R}x, H^1(V, \mathfrak{g})),$$
$$E_2^{0,2} = H^0(\mathbb{R}x, H^2(V, \mathfrak{g})) = \mathrm{Inv}_x\, H^2(V, \mathfrak{g}).$$

Proof. Since the Lie algebra $\mathbb{R}x$ is one-dimensional, we see that $E_2^{p,q} = \{0\}$ for all $p > 1$. Therefore, the differential

$$d_2^{p,q} \colon E_2^{p,q} \to E_2^{p+2,q-1}$$

is trivial and the spectral sequence is stabilized in the second term. □

The computation of $H^2(\mathfrak{g}_-, \mathfrak{g})$ can be reduced essentially to the decomposition of $\mathfrak{sl}(2,\mathbb{R})$-modules $H^1(V,\mathfrak{g})$ and $H^2(V,\mathfrak{g})$ into sums of irreducible submodules.

Lemma 1. *The space $H^k(\mathbb{R}x, V_p)$ is trivial for $k \geq 2$ and is one-dimensional for $k = 0, 1$.*

Let v_0 and v_p be the highest and the lowest vectors of V_p (that is $h.v_0 = pv_0$ and $h.v_p = -pv_p$). Then $H^0(\mathbb{R}x, V_p)$ is generated by v_0, and $H^1(\mathbb{R}x, V_p)$ is generated by $[\alpha\colon x \to v_p]$.

Proof. Immediately follows from the explicit description of the structure of irreducible $\mathfrak{sl}(2,\mathbb{R})$-modules. □

Let us identify $\mathfrak{a}$ with the subalgebra of $\mathfrak{gl}(V)$ corresponding to the action of $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{gl}(m,\mathbb{R})$ on V. Then the cohomology spaces $H^k(V,\mathfrak{g})$ can be described via the classical Spencer cohomology spaces determined by the subalgebra $\mathfrak{a} \subset \mathfrak{gl}(V)$. Recall that the Spencer operator S^k is defined as:

$$S^k\colon \operatorname{Hom}(\wedge^k V, \mathfrak{a}) \to \operatorname{Hom}(\wedge^{k+1} V, V),$$

$$S^k(\phi)(v_1 \wedge v_2 \wedge \cdots \wedge v_{k+1}) = \sum_{i=1}^{k+1} (-1)^i \phi(v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{k+1}) v_i.$$

Lemma 2. *We have $H^0(V,\mathfrak{g}) = V$ and*

$$H^k(V,\mathfrak{g}) = \ker S^k \oplus \operatorname{Hom}(\wedge^k V, V)/\operatorname{im} S^{k-1}$$

for all $k \geq 1$.

Proof. Indeed, let us represent an arbitrary cocycle $c \in C^k(V,\mathfrak{g})$ as $c = c_{\mathfrak{a}} + c_V$, where $c_{\mathfrak{a}} \in \operatorname{Hom}(\wedge^k V, \mathfrak{a})$ and $c_V \in \operatorname{Hom}(\wedge^k V, V)$. Since V is commutative Lie algebra, we have

$$(\partial c) = S^k(c_{\mathfrak{a}}) \in \operatorname{Hom}(\wedge^{k+1} V, V).$$

This immediately implies the statement of the lemma. □

For $k = 1, 2$ the mappings S^k can be described explicitly.

Lemma 3.

The operator S^1 is injective if $m \geq 2$ and $k \geq 2$.

Proof. Let us note that $\ker S^1$ is precisely the first prolongation $\mathfrak{a}^{(1)}$ of the subalgebra $\mathfrak{a} \subset \mathfrak{gl}(V)$. Suppose that $\mathfrak{a}^{(1)} \neq \{0\}$. Then the algebra $V + \mathfrak{a} + \sum_{i=1}^{\infty} \mathfrak{a}^{(i)}$ is an irreducible graded Lie algebra of the order ≥ 2 (see [11]). Then from [11, Lemma 7.3] it follows that the difference between the highest and the lowest roots of $\mathfrak{a}$-module V is equal to the sum of the highest roots of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{gl}(m,\mathbb{R})$. This is not possible and therefore $\ker S^1 = \mathfrak{a}^{(1)} = \{0\}$. □

Lemma 4. *The operator $\mathcal{S}^2$ is injective for $m \geq 3, k \geq 3$ and $m = 2, k \geq 4$.*

Proof. First we prove that $\ker \mathcal{S}^2 = 0$ for $m \geq 3, k \geq 3$.

Let α be an arbitrary element of $\ker \mathcal{S}^2$. Put

$$\alpha_{ij}(m_1, m_2) = \alpha(v^i \otimes m_1, v^j \otimes m_2) \in \mathfrak{a}.$$

Let us show that $\alpha_{ij} = 0$ for all $i, j \geq 2$. Indeed, we have

$$\alpha_{ij}(m_1, m_2)v^0 \otimes m_3 - \alpha_{0j}(m_3, m_2)v^i \otimes m_1 + \alpha_{0i}(m_3, m_1)v^j \otimes m_2 = 0.$$

But for any element $X \in \mathfrak{a}$

$$Xv^i \otimes m \in \langle v^{i-1} \otimes m, v^i \otimes W, v^{i+1} \otimes m\rangle.$$

Hence, $\alpha_{ij}(m_1, m_2)v^0 \otimes m_3 = 0$ for every vector m_3 which is not laying in the linear span of m_1 and m_2. Therefore

$$\alpha_{ij}(m_1, m_2) \in \langle x, h, z\rangle, \tag{5}$$

where z is a center element of the Lie algebra $\mathfrak{gl}(m,\mathbb{R})$. Similarly,

$$\alpha_{ij}(m_1, m_2)v^1 \otimes m_3 - \alpha_{1j}(m_3, m_2)v^i \otimes m_1 + \alpha_{1i}(m_3, m_1)v^j \otimes m_2 = 0.$$

But from (5) we see that

$$\alpha_{ij}(m_1, m_2)v^1 \otimes m_3 \in (\mathbb{R}v^0 \oplus \mathbb{R}v^1) \otimes m_3.$$

Therefore, $\alpha_{ij}(m_1, m_2)v^1 \otimes m_3 = 0$ for every vector m_3 which is not laying in the linear span of m_1 and m_2. This is possible only if $\alpha_{ij} = 0$. In the same way we can prove that $\alpha_{ij} = 0$ for all $i, j \leq k - 2$.

Consider now the following subspace $Q \subset \wedge^2 V$:

$$Q = \{w \in \wedge^2 V \mid \alpha(w) = 0, \quad \alpha \in \ker \mathcal{S}^2\}.$$

It is clear that Q is a submodule of the $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{gl}(m,\mathbb{R})$-module $\wedge^2 V$. As we have just proved, $v^i \otimes W \wedge v^j \otimes W \subset Q$ for all pairs of i and j such that $i, j \geq 2$ or $i, j \leq k-2$. Hence, Q contains also the submodule generated by these elements.

We state that these elements generate whole $\wedge^2 V$. It is sufficient to prove that if $v^i \otimes W \wedge v^j \otimes W \subset Q$ for all pairs of i and j such that $i, j \geq l + 1$ and $i, j \leq l$ then $v^i \otimes W \wedge v^j \otimes W \subset Q$ for all pairs of i and j such that $i, j \geq l - 1$ and $i, j \leq l + 1$. First, consider an element $v^{l+1} \otimes m_1 \wedge v^{l+j} \otimes m_2$ where $j \geq 2$. Then after the action of x on this element we get that $v^l \otimes m_1 \wedge v^{l+j} \otimes m_2 \in Q$. Similarly, using the action of the element y on $v^l \otimes m_1 \wedge v^{l-j+1} \otimes m_2$ we get $v^{l+1} \otimes m_1 \wedge v^{l-j+1} \otimes m_2 \in Q$.

The only one type of elements for which we don't know yet if they belong to Q are elements of the form $v^l \otimes m_1 \wedge v^{l+1} \otimes m_2$. If we act by the element x on $v^l \otimes m_1 \wedge v^{l+2} \otimes m_2$ we obtain

$$v^{l-1} \otimes m_1 \wedge v^{l+2} \otimes m_2 + v^l \otimes m_1 \wedge v^{l+1} \otimes m_2 \in Q.$$

On the other hand, after the action of the element y on $v^{l-1} \otimes m_1 \wedge v^{l+1} \otimes m_2$ we get

$$(k-i-1)v^{l-1} \otimes m_1 \wedge v^{l+2} \otimes m_2 + (k-i+1)v^l \otimes m_1 \wedge v^{l+1} \otimes m_2 \in Q.$$

Therefore elements $v^{l-1} \otimes m_1 \wedge v^{l+2} \otimes m_2$ and $v^l \otimes m_1 \wedge v^{l-1} \otimes m_2$ belong to Q.

Now, consider the case $m = 2$. Using the same reasoning with the maps α_{ij} one can show that $\alpha_{ij} = 0$ if $i, j \geq 3$ or $i, j \leq k-3$. Therefore if $k \geq 6$ the map $\mathcal{S}^2$ is injective. Direct computation shows that $\ker \mathcal{S}^2 = 0$ for $k = 5$ and $k = 4$. □

Lemma 5. *If $m = 2, k = 3$ then a kernel of the operator $\mathcal{S}^2$ is a one dimensional space.*

Proof. Can be shown by direct computation. □

3.1. **Cohomology of contact Lie algebras of higher order.** As shown in [13, 5], the Tanaka symbol of the contact distribution on $J^k(\mathbb{R}, \mathbb{R}^m)$, can be identified with the Lie algebra of polynomial vector fields on $\mathbb{R}^{m+1}$, equipped with a grading that depends on k. More precisely, let U be a one-dimensional vector space spanned by x and W be an m-dimensional vector space spanned by $v^0 \otimes e_i$, $i = 1, \dots, m$. In particular, U has degree -1, and elements of W have degree $-k-1$. Then the symbol algebra of the contact distribution on $J^k(\mathbb{R}, \mathbb{R}^m)$ is

$$\bar{\mathfrak{g}} = S(U^* \oplus W^*) \otimes (U \oplus W).$$

The Lie algebra $\mathfrak{g}$ is, obviously, included into the Lie algebra $\bar{\mathfrak{g}}$. Additionally we should note that $\mathfrak{g}_- = \bar{\mathfrak{g}}_-$. The inclusion above induces the map between cohomologies. We are going to study this map in more detail.

Let $V = \oplus_{i=0}^k S^i(U^*) \otimes W = V_k \otimes W$ and

$$F = \oplus_{i=1}^k S^i(U^*) \otimes W = \langle v^1, \dots, v^k \rangle \otimes W.$$

We define an operator

$$\delta \colon \operatorname{Hom}(\wedge^p F, U) \to \operatorname{Hom}(\wedge^{p+1} F, V/F) = \operatorname{Hom}(\wedge^{p+1} F, W)$$

by the following rule

$$(6) \quad (\delta\omega)(A_1, \dots, A_{p+1}) = \left(\sum_{i=1}^{p+1} (-1)^i \omega(A_1, \dots, \hat{A}_i, \dots, A_{p+1}).A_i \right) / F.$$

In this setting the structure of the cohomology space $H(\bar{\mathfrak{g}}_-, \bar{\mathfrak{g}})$ is described by the following result of Morimoto [13].

Theorem (Morimoto)**.** *One has the following long exact sequence:*

$$\cdots \to \operatorname{Hom}(\wedge^p F, W) \to H^p(\bar{\mathfrak{g}}_-, \bar{\mathfrak{g}}) \\ \to \operatorname{Hom}(\wedge^p F, U) \xrightarrow{\delta} \operatorname{Hom}(\wedge^{p+1} F, W) \to \cdots .$$

As an immediate corollary of this theorem we obtain that $H^1(\bar{\mathfrak{g}}_-, \bar{\mathfrak{g}}) = 0$ for $m \geq 2$. Thus, we have the following exact sequence:

$$0 \to \operatorname{Hom}(F, U) \xrightarrow{\delta} \operatorname{Hom}(\wedge^2 F, W) \to H^2(\bar{\mathfrak{g}}_-, \bar{\mathfrak{g}}) \\ \to \operatorname{Hom}(\wedge^2 F, U) \xrightarrow{\delta} \operatorname{Hom}(\wedge^3 F, W).$$

We see that $H^2(\bar{\mathfrak{g}}_-, \bar{\mathfrak{g}})$ consists of 2 parts:

$$\operatorname{Hom}(\wedge^2 F, W)/\delta \operatorname{Hom}(F, U)$$

and

$$\ker\left(\delta\colon \operatorname{Hom}(\wedge^2 F, U) \to \operatorname{Hom}(\wedge^3 F, W)\right).$$

Consider now a map

$$\gamma\colon H^2(\mathfrak{g}_-, \mathfrak{g}) \to H^2(\bar{\mathfrak{g}}_-, \bar{\mathfrak{g}})$$

which is induced by the inclusion of Lie algebras $\mathfrak{g} \hookrightarrow \bar{\mathfrak{g}}$. Geometrically, it maps the fundamental invariants of the geometry with the symbol $\mathfrak{g}$ to the fundamental invariants of the geometry with the underlying distribution with the Tanaka symbol equal to $\mathfrak{g}_- = \bar{\mathfrak{g}}_-$. However, in case of the geometry defined by system (3) the underlying distribution is exactly the model distribution with this symbol. Thus, we get the following result.

Theorem 1. *The cohomology class of the curvature function of any regular connection associated with a system of ODEs is killed by γ.*

We give the proof of this Theorem in Subsection 4.2 using the parametric formulas for the curvature function of regular connections associated with (3).

In the next subsection we are going to compute cohomology the part of cohomology $H^2_+(\mathfrak{g}_-, \mathfrak{g})$ which lays in the kernel of the map γ.

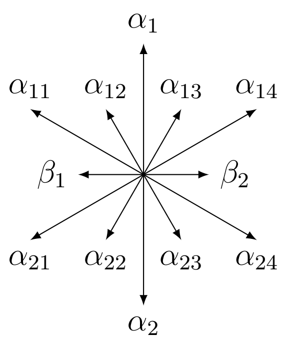
3.2. **The structure of the space $E_2^{0,2}$.** We are interested in the kernel of the restriction of the map γ to the space $E_2^{0,2} = \operatorname{Inv}_x H^2(V, \mathfrak{g})$. As we know

$$H^2(V, \mathfrak{g}) = \frac{\ker \mathcal{S}^2 \oplus \operatorname{Hom}(\wedge^2 V, V)}{\operatorname{im} S^1}.$$

Since the kernel of the map $\mathcal{S}^2$ is non-zero only in the case $m = 2, k = 3$ first we consider this case.

Lemma 6. *For $m = 2, k = 3$ the map $\gamma\colon \ker \mathcal{S}^2 \to H^2(\bar{\mathfrak{g}}_-, \bar{\mathfrak{g}})$ is injective.*

Proof. The generator of $\ker \mathcal{S}^2$ can be represented as a Lie bracket in the special Lie algebra of type G_2. We define its root system by the following picture



and associate an element $v^i \otimes e_j$ with the element from $\mathfrak{g}_{\alpha_{ji}}$ and elements x, y, e_1^2, e_2^1 with elements from $\mathfrak{g}_{\beta_1}, \mathfrak{g}_{\beta_2}, \mathfrak{g}_{\alpha_1}, \mathfrak{g}_{\alpha_2}$ respectively. The fact that the map defined by the Lie bracket in G_2 is in the kernel of $\mathcal{S}^2$ follows from the Jacobi identity.

Now it is easy to see that $\mathrm{Hom}(\wedge^2 F, U)$ component of this map sends $v^1 \otimes e_1 \wedge v^1 \otimes e_2$ to $\mathbb{R}x$ and this part belongs to the $\ker \delta$. Therefore, $\gamma(\alpha) \neq 0$ for any non-zero element $\alpha \in \ker \mathcal{S}^2$.

$\square$

Define the canonical $\mathfrak{h}$-invariant morphisms $i_F\colon F \to V$, $i_{\wedge^2 F}\colon \wedge^2 F \to \wedge^2 V$ and $\pi_W\colon V \to V/F$. Further, define

$$\alpha\colon \mathrm{Hom}(\wedge^2 V, V) \to \mathrm{Hom}(\wedge^2 F, V/F)$$

as the composition of $i^*_{\wedge^2 F}$ and π_W. The map α sends every morphism $c \in \mathrm{Hom}(\wedge^2 V, V)$ to a morphism

$$\alpha(c) = \pi_W \circ c \circ i_{\wedge^2 F}.$$

The kernel of the map α is an $\mathfrak{h}$-invariant submodule, since α is $\mathfrak{h}$-invariant.

The explicit description of the kernel of γ in the general case is based on several lemmas.

Lemma 7. *The restriction π_W^x of the map π_W to $\mathrm{Inv}_x \mathrm{Hom}(\wedge^2 V, V)$ is injective. Moreover the image of the map π_W^x is equal to*

$$\ker x^{k+1}|_{\mathrm{Hom}(\wedge^2 V, W)}.$$

Proof. Indeed, suppose $\alpha \in \operatorname{Inv}_x \operatorname{Hom}(\wedge^2 V, V)$. Then we have

$$x.\sum_{i=0}^{k} \alpha_i e_i = \sum_{i=0}^{k} (x.\alpha_i) e_i + \sum_{i=0}^{k-1} \alpha_{i+1} e_i = 0.$$

Hence, $\alpha_i = -x.\alpha_{i-1}$ for all $i = 1, \dots, k$. Therefore, $\alpha_0 = 0$ implies that $\alpha_i = 0$ for all $i > 0$ and, thus, $\alpha = 0$.

The second part of the lemma follows directly from the equality

$$x.\alpha_n = (-1)^{k+1} x^{k+1}.\alpha_0 = 0.$$

□

Recall, that in the section 3.1 we defined the operator

$$\delta\colon \operatorname{Hom}(\wedge^p F, U) \to \operatorname{Hom}(\wedge^{p+1} F, V/F) = \operatorname{Hom}(\wedge^{p+1} F, W)$$

by the formula (6). This operator plays a crucial role in the prove of the following theorem.

Theorem 2. *For $k \geq 3$ the $E_2^{0,2}$ part of the space $\{c \in H^2(\mathfrak{g}_-, \mathfrak{g}) \mid \gamma(c) \in \operatorname{im} \delta\}$ is isomorphic (as a $\mathfrak{gl}(m, \mathbb{R})$-module) to the space $\operatorname{tr}_0 S^2(W^*) \otimes W$, where tr_0 means a traceless part. All elements of the space have degree 2.*

Proof. Let $\bar{\alpha}\colon \operatorname{Hom}(V, \mathfrak{sl}(2, \mathbb{R})) \to \operatorname{Hom}(F, \mathfrak{sl}(2, \mathbb{R})/\langle h, y\rangle) = \operatorname{Hom}(F, U)$ be the canonical projection. Then the following diagram is commutative:

$$\begin{array}{ccc} \operatorname{Hom}(V, \mathfrak{sl}(2, \mathbb{R})) & \xrightarrow{S^1} & \operatorname{Hom}(\wedge^2 V, V) \\ \big\downarrow{\scriptstyle \bar{\alpha}} & & \big\downarrow{\scriptstyle \alpha} \\ \operatorname{Hom}(F, U) & \xrightarrow{\delta} & \operatorname{Hom}(\wedge^2 F, W). \end{array} \tag{7}$$

Therefore the space $\{c \in \operatorname{Inv}_x \operatorname{Hom}(\wedge^2 V, V) \mid \gamma(c) \in \operatorname{im} \delta\}$ is equal to the set of x-invariant elements in $\ker \alpha + \operatorname{im} S^1$. By Lemma 7 we need to compute the space:

$$\begin{aligned} T &= \operatorname{Inv}_{x^{k+1}} \pi_W \left(\ker \alpha + \operatorname{im} S^1\right) \\ &= \operatorname{Inv}_{x^{k+1}} \left(\operatorname{Hom}(\wedge^2 W + W \otimes F, W) + \pi_W \operatorname{im} S^1\right). \end{aligned}$$

All elements from $\operatorname{Hom}(\wedge^2 W + W \otimes F, W) + \operatorname{im} S^1$ have the following form:

$$v = v^{0*} e_{i_1}^* \wedge v^{i*} e_{i_2}^* \otimes A_i^{i_1, i_2} + v^{1*} e_{i_1}^* \wedge v^{i-1*} e_{i_2}^* \otimes \beta_i^{i_2} e_{i_1}.$$

Then the action of x^{k+1} on v is

$$x^{k+1}.v = (-1)^{k+1}\sum_{j=0}^{k+1} C_{k+1}^{j} v^{j^*} e_{i_1}^* \wedge v^{i+k+1-j^*} e_{i_2}^* \otimes A_i^{i_1,i_2}$$

$$+\sum_{j=1}^{k+1} C_{k+1}^{j-1} v^{j^*} e_{i_1}^* \wedge v^{i+k+1-j^*} e_{i_2}^* \otimes \beta_i^{i_2} e_{i_1}$$

Decompose the right hand side in the standard basis of $\wedge^2 V^*$. Then from $x^{k+1}.v = 0$ we get:

- for $i < j < \frac{i+k+1}{2}$ a coefficient at $v^{j^*} e_{i_1}^* \wedge v^{i+k+1-j^*} e_{i_2}^*$ gives:

(8) $$C_{k+1}^{j} A_i^{i_1,i_2} - C_{k+1}^{i+k+1-j} A_i^{i_2,i_1} + C_{k+1}^{j-1} \beta_i^{i_2} e_{i_1} - C_{k+1}^{i+k-j} \beta_i^{i_1} e_{i_2} = 0;$$

- for $j = \frac{i+k+1}{2}$ a coefficient at $v^{j^*} e_{i_1}^* \wedge v^{j^*} e_{i_2}^*$ gives:

(9) $$C_{k+1}^{j}(A_i^{i_1,i_2} - A_i^{i_2,i_1}) + C_{k+1}^{j-1}(\beta_i^{i_2} e_{i_1} - \beta_i^{i_1} e_{i_2}) = 0;$$

Equations (8-9) imply that for every β_i, $i < k-1$ there exists at most one tensor A_i which satisfies equations (8-9). Moreover if $\beta_i = 0$ then A_i should be zero. Using equation (9) we conclude the same for β_{k-1} and for the antisymmetric part of the tensor A_{k-1}.

This can be reformulated as follows. The space $\wedge^2 V^*$ is decomposed into the direct sum

$$\wedge^2 V^* = \wedge^2 V_k^* \otimes S^2(W^*) + S^2(V_k^*) \otimes \wedge^2 W^*.$$

Using this decomposition, we can check directly that elements from $v^{0^*} \otimes v^{k^*} \otimes W^* \otimes W^* \otimes W$ and $\langle v^0 \wedge v^{k-1}\rangle^* \otimes S^2 W^* \otimes W$ are x^{k+1}-invariant and belong to $\ker\alpha$. Then the space

$$I = \mathrm{Inv}_{x^{k+1}}\, \pi_W \left(\frac{\ker\alpha + \operatorname{im} S^1}{\operatorname{im} S^1} \right)$$

belongs to

$$T = v^{0^*} \otimes v^{k^*} \otimes W^* \otimes W^* \otimes W + \langle v^0 \wedge v^{k-1}\rangle^* \otimes S^2 W^* \otimes W.$$

It remains to factor the space T by $T \cap \pi_W(\operatorname{im} S^1)$. Note that $\pi_W S^1\left(\langle v^{k^*}\rangle \otimes W^* \otimes \mathfrak{gl}(m,\mathbb{R})\right)$ is exactly

$$v^{0^*} \otimes v^{k^*} \otimes W^* \otimes W^* \otimes W.$$

Therefore

$$I \subset \langle v^0 \wedge v^{k-1}\rangle^* \otimes S^2 W^* \otimes W$$

and the space I has the degree 2. The only subspace in $T \cap \pi_W(\operatorname{im} S^1)$ which has degree 2 is the space generated by

$$\omega_{e_i} = 2(v^{k-2} \otimes e_i)^* \otimes x + (k-1)(v^{k-1} \otimes e_i)^* \otimes h + k(k-1)(v^k \otimes e_i)^* \otimes y.$$

Elements ω_{e_i} allows us to normalize trace of an element from $\langle v^0 \wedge v^{k-1}\rangle^* \otimes S^2W^* \otimes W$ to be equal to 0 since

$$\pi_W(S^1(v^{k-1^*} \otimes e_i^* \otimes h)) = 2v^{0^*} \wedge v^{k-1^*} \otimes \sum_j e_i^* \odot e_j^* \otimes e_j.$$

□

The explicit formulas for the corresponding fundamental invariant are given in Theorem 4.

3.3. **The structure of the space $E_2^{1,1}$.** As shown in [6] for the case of a scalar ODE, this part of the cohomology space corresponds to generalized Wilczynski invariants for the systems of ODEs. For completeness we provide a purely algebraic description of this space.

According to the Proposition 3, the space $E_2^{1,1}$ is isomorphic to

$$H^1(\mathbb{R}x, \mathrm{Hom}(V, V))/\operatorname{im}\mathcal{S}^0,$$

where $\mathcal{S}^0$ is an inclusion of Lie algebra $\mathfrak{a}$ into $\mathfrak{gl}(V)$. Lemma 2 implies that we should describe the set of y-invariant elements in $\mathfrak{sl}(2,\mathbb{R})$-module $\mathfrak{gl}(V)/\mathfrak{a}$.

The $\mathfrak{a}$-module $\mathfrak{gl}(V)$ is isomorphic to $\mathfrak{gl}(V_k) \otimes \mathfrak{gl}(W)$. Let's identify Lie algebra $\mathfrak{a}$ with its image in $\mathfrak{gl}(V)$. In particular, we denote as y the image of $y \in \mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{gl}(V)$.

Theorem 3. *The space of Y-invariant elements in the $\mathfrak{a}$-module $\mathfrak{gl}(V)/\mathfrak{a}$ is the sum of the following $\mathfrak{gl}(m,\mathbb{R})$-modules:*

$$A_2 = \mathbb{R}y \otimes \mathfrak{sl}(W)$$
$$A_{i+1} = \mathbb{R}y^i \otimes \mathfrak{gl}(W), \quad i = 2, \dots, k$$

For every element $\phi \in A_i$ the corresponding element $c_\phi \in E_2^{1,1}$ of the form $c_\phi \colon x \to \phi$ has degree i.

Proof. The decomposition of the $\mathfrak{sl}(2,\mathbb{R})$-module $\mathfrak{gl}(V_k)$ is well known:

$$\mathfrak{gl}(V_k) = V_0 \oplus V_2 \oplus \dots \oplus V_{2k}.$$

Note that all endomorphisms y^i, $i = 0, \dots, k$ are y-invariant and linearly independent. Therefore the space of y-invariant elements is the sum of submodules $\mathbb{R}y^i \otimes \mathfrak{gl}(W)$ for $i = 0, \dots, k$.

The space of y-invariant elements in $\mathfrak{a}$ is nothing else but $\mathbb{R}y \oplus \mathfrak{gl}(m,\mathbb{R})$. It is not hard to see that under the natural inclusion $\mathfrak{a} \hookrightarrow \mathfrak{gl}(V)$ the space $\mathfrak{gl}(m,\mathbb{R})$ goes to $\mathbb{R}y^0 \otimes \mathfrak{gl}(W)$ and $\mathbb{R}y$ goes to $\mathbb{R}y \otimes \mathrm{Id}_W$. Finally, we can identify the space $\mathbb{R}y \otimes \mathfrak{gl}(W)/\mathbb{R}y \otimes \mathrm{Id}_W$ with $\mathbb{R}y \otimes \mathfrak{sl}(W)$. □

Each of the submodules A_i, $i = 2, \dots, k$ corresponds to the generalized Wilczynski invariant W_i of degree i.

4. Parametric computation of the regular Cartan connections

The main aim of this section is to provide an explicit formula for the invariant I_2, which is described by Theorem 2 and to provie the proof of Theorem 1.

For every regular Cartan connection adapted to the equation (3) we can choose a section $s\colon \mathcal{E} \to \mathcal{G}$ such that the pullback of the connection form to $\mathcal{E}$ is:

$$\omega = \sum_{r=1}^{k+1} \omega^i_{-r} v^{k+1-r} \otimes e_i + \omega_x x + \omega_h h + \omega^i_j e^j_i + \omega_y y,$$

where

$$\begin{aligned}
\omega^i_{-r} &= \theta^i_{-r} + \sum_{s=r+1}^{k+1} A^{i,-s}_{j,-r} \theta^j_{-s}, \\
\omega_x &= -\theta_x + \sum_{s=2}^{k+1} B^{-s}_j \theta^j_{-s}, \\
\omega_h &= \sum_{s=1}^{k+1} C^{-s}_j \theta^j_{-s}, \\
\omega^i_j &= D^{i,x}_j \theta_x + \sum_{s=1}^{k+1} D^{i,-s}_{j,l} \theta^l_{-s}, \\
\omega_y &= E^x \theta_x + \sum_{s=1}^{k+1} E^{-s}_j \theta^j_{-s}.
\end{aligned}$$

and the fors θ_x, θ^i_{-r} are given by:

$$\begin{aligned}
\theta_x &= dx, \\
\theta^i_{-1} &= dy^i_k - f^i\, dx, \\
\theta^i_{-r} &= dy^i_{k+1-r} - y^i_{k+2-r}\, dx, \quad 2 \le r \le k+1.
\end{aligned}$$

Let Ω be the curvature tensor of ω. We use the following notation:

$$\Omega = \sum_{r=1}^{k+1} \Omega^i_{-r} v^{k+1-r} \otimes e_i + \Omega_x x + \Omega_h h + \Omega^j_i e^i_j + \Omega_y y$$

4.1. **Computation of I_2.** . To compute I_2 it is sufficient to compute the curvature of the normal Cartan connection up to degree 2.

There are no fundamental invariants in degree 1. So, all coefficients of Ω in this degree should vanish. We have:

$$\begin{aligned}
\Omega_x =\ & d\omega_x + 2\omega_h \wedge \omega_x \equiv B_j^{-2}\theta_x \wedge \theta^j_{-1} \\
& + 2C_j^{-2}\theta^j_{-1} \wedge \theta_x \quad \operatorname{mod} \left\langle \omega^p_{-2}, \dots, \omega^p_{-k-1} \right\rangle, \\
\Omega^i_{-1} =\ & d\omega^i_{-1} - k\omega_h \wedge \omega^i_{-1} + k\omega_y \wedge \omega^i_{-2} + \omega^i_j \wedge \omega^j_{-1} \\
& \equiv \frac{\partial f^i}{\partial y^j_k}\theta_x \wedge \theta^j_{-1} + A^{i,-2}_{j,-1}\theta_x \wedge \theta^j_{-1} - kC_l^{-1}\theta^l_{-1} \wedge \theta^i_{-1} \\
& + D^{i,x}_j \theta_x \wedge \theta^j_{-1} + D^{i,-1}_{j,l}\theta^l_{-1} \wedge \theta^j_{-1} \quad \operatorname{mod} \left\langle \omega^p_{-2}, \dots, \omega^p_{-k-1} \right\rangle, \\
\Omega^i_{-r} =\ & d\omega^i_{-r} + \omega_x \wedge \omega^i_{-r+1} + (2r-k-2)\omega_h \wedge \omega^i_{-r} \\
& + (k+1-r)r\, \omega_y \wedge \omega^i_{-r-1} + \omega^i_j \wedge \omega^j_{-r} \\
& \equiv (A^{i,-r-1}_{j,-r} - A^{i,-r}_{j,-r+1} + D^{i,x}_j)\theta_x \wedge \theta^j_{-r} \\
& + (2r-k-2)C_l^{-1}\theta^l_{-1} \wedge \theta^i_{-r} + D^{i,-1}_{j,l}\theta^l_{-1} \wedge \theta^j_{-r} \\
& \quad \operatorname{mod} \left\langle \omega^{p_1}_{-s_1} \wedge \omega^{p_1}_{-s_1} \mid s_1 + s_2 > r+1 \right\rangle, \quad r = 2, \dots, k, \\
\Omega^i_{-k-1} =\ & d\omega^i_{-k-1} + \omega_x \wedge \omega^i_{-k} + k\omega_h \wedge \omega^i_{-k-1} + \omega^i_j \wedge \omega^j_{-k-1} \\
& \equiv (-A^{i,-k-1}_{j,-k} + D^{i,x}_j)\theta_x \wedge \theta^j_{-k-1} + kC_l^{-1}\theta^l_{-1} \wedge \theta^i_{-k-1} + \\
& D^{i,-1}_{j,l}\theta^l_{-1} \wedge \theta^j_{-k-1} \quad \operatorname{mod} \left\langle \omega^{p_1}_{-s_1} \wedge \omega^{p_1}_{-s_1} \mid s_1 + s_2 > k+2 \right\rangle.
\end{aligned}$$

Setting the terms above equal to 0 we get:

$$\begin{aligned}
& C_j^{-1} = B_j^{-2} = D^{i,-1}_{j,l} = 0, \\
& D^{i,x}_j = \frac{1}{k+1}\frac{\partial f^i}{\partial y^j_k}, \quad A^{i,-r}_{-r-1} = \frac{(k+1-r)}{k+1}\frac{\partial f^i}{\partial y^j_k},
\end{aligned}$$

where $r = 1, \dots, k$.

Now proceed to the second degree. We consider only $E_2^{0,2}$ part of the curvature. This means that we want to compute only the coefficients of Ω^i_{-r} at $\omega^i_{-r_1} \wedge \omega^j_{-r_2}$. We have:

$$\begin{aligned}
\Omega^i_{-1} \equiv & \left(-kC_j^{-2}\delta^i_l + \frac{\partial A^{i,-2}_{j,-1}}{\partial y^l_k} + kE_j^{-1}\delta^i_l + D^{i,-2}_{j,l} \right) \theta^j_{-2} \wedge \theta^i_{-1} \\
& \operatorname{mod} \left\langle \omega_x, \omega^{p_1}_{-2} \wedge \omega^{p_2}_{-2}, \omega^p_{-3}, \dots, \omega^p_{-k-1} \right\rangle, \\
\Omega^i_{-k} \equiv & \left((k-2)C_j^{-2}\delta^i_l + \frac{\partial A^{i,-k-1}_{j,-k}}{\partial y^l_k} + kE_j^{-1}\delta^i_l + D^{i,-2}_{j,l} \right) \theta^j_{-k} \wedge \theta^i_{-1}
\end{aligned}$$

$$\mod \left\langle \omega_x, \omega^{p_1}_{-s_1} \wedge \omega^{p_1}_{-s_1} \,|\, s_1 + s_2 > k+2 \right\rangle,$$

$$\Omega^i_{-k-1} \equiv \left(kC_j^{-2}\delta^i_l + D^{i,-2}_{j,l}\right)\theta^j_{-k-1} \wedge \theta^i_{-1}$$

$$\mod \left\langle \omega_x, \omega^{p_1}_{-s_1} \wedge \omega^{p_1}_{-s_1} \,|\, s_1 + s_2 > k+2 \right\rangle.$$

The normality condition in degree 2 implies that the Ω^i_{-k-1} term is equal to 0 and terms of Ω^i_{-1} and Ω^i_{-k} are trace free. As a result we obtain that $E^{-1}_j = 0$, $D^{j,-2}_{i,l} = 0$ if $l \neq j$ and $D^i_{j,i} = kC_j = \frac{1}{2}\operatorname{tr}\frac{\partial^2 f^i}{\partial y^j_k \partial y^l_k}$. The remaining non-zero part of the tensors above gives us exactly the required invariant of the second degree prescribed by Theorem 2:

$$(I_2)^i_{jl} = \operatorname{tr}_0 \frac{\partial^2 f^i}{\partial y^j_k \partial y^l_k}.$$

We summarize the computations of this section in the following theorem.

Theorem 4. *For the system of ODEs* (3) *of the order* ≥ 3 *the following symmetric tensor is a relative differential invariant of the second degree:*

$$(I_2)^i_{jl} = \operatorname{tr}_0 \frac{\partial^2 f^i}{\partial y^j_k \partial y^l_k},$$

where tr_0 *is the trace-free part of the tensor.*

4.2. **Proof of Theorem 1.** Let, as above, ω be the pullback of an arbitrary regular Cartan connection associated with system of ODEs (3) to the equation manifold $\mathcal{E}$. Let Ω be its curvature and c be its curvature function.

In notation of Subsection 3.1 the part $\operatorname{Hom}(\wedge^2 F, W)$ of the curvature function c corresponds to the coefficients of Ω^i_{-k-1} at $\omega^{j_1}_{-s_1} \wedge \omega^{j_2}_{-s_2}$, $s_1, s_2 = 2, \dots, k+1$. Using the above formula for Ω^i_{-k-1} we get:

$$\Omega^i_{-k-1} = d\omega^i_{-k-1} + \omega_x \wedge \omega^i_{-k} \mod \langle \omega^p_{-k-1} \rangle =$$
$$d\theta^i_{-k-1} + \omega_x \wedge \omega^i_{-k} = \theta_x \wedge \theta^i_{-k} + \omega_x \wedge \omega^i_{-k}.$$

Using the fact that $\omega^i_{-k} = \theta^i_{-k} \mod \langle \omega^p_{-k-1} \rangle$ and that

$$\theta_x = -\omega_x + \sum_{s=2}^{k+1} \bar{B}^{-s}_j \omega^j_{-s}$$

for some functions $\bar{B}_j^{-s}$ (expressed polynomially through B_j^{-s} and $A_{j,-r}^{i,-s}$), we further get:

$$\Omega_{-k-1}^i = (-\omega_x + \sum_{s=2}^{k+1} \bar{B}_j^{-s}\omega_{-s}^j) \wedge \omega_{-k}^i + \omega_x \wedge \omega_{-k}^i = \sum_{s=2}^{k} \bar{B}_j^{-s}\omega_{-s}^j \wedge \omega_{-k}^i \mod \langle \omega_{-k-1}^p \rangle. \tag{10}$$

Define the map $\alpha \in \mathrm{Hom}(F, U)$ by:

$$\alpha \colon v^{k+1-s} \otimes e_i \mapsto -\bar{B}_j^{-s} x, \quad s = 2, \dots, k+1,$$
$$\alpha \colon v^k \otimes e_i \mapsto 0.$$

Then equation (10) means that the $\mathrm{Hom}(\wedge^2 F, W)$-part of the structure function c is equal exactly to $\delta(\alpha)$. In particular, we see that the component of $\gamma(c)$ lying in $\mathrm{Hom}(\wedge^2 F, W)/\delta\,(\mathrm{Hom}(F, U))$ vanishes identically for any regular Cartan connection associated with system (3).

To prove that the other part of $\delta(c)$ lying in:

$$\ker\big(\delta\colon\ \mathrm{Hom}(\wedge^2 F, U) \to \mathrm{Hom}(\wedge^3 F, W)\big)$$

vanishes identically, we need to prove that $\delta(c') = 0$, where c' is the $\mathrm{Hom}(\wedge^2 F, U)$-part of the structure function c. This is done in the same way by analyzing the coefficients of Ω_x at $\omega_{-s_1}^{j_1} \wedge \omega_{-s_2}^{j_2}$, $s_1, s_2 = 2, \dots, k+1$ in the right hand side of the structure equation $d\Omega = [\Omega, \omega]$, in particular, of the expression for $d\Omega_{-k-1}^i$.

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