Nonstandard operators in Grassmannian geometry

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Abstract

We present a construction of curved analogues of the nonstandard operators on Grassmannians parallel to the construction of the Paneitz operator in [3], but technically more demanding. In particular, the construction breaks down in the presence of torsion. In the second part, we prove that the nonstandard operators are not strongly invariant.

1 Introduction

The curved Casimir operators for parabolic geometries were originally introduced in [2]. As described in that paper, their strong naturality properties together with the fact that they act by a multiplication by a scalar on irreducible bundles enable us to use them to construct higher order invariant operators. Concrete examples of such constructions in conformal geometry are discussed in [3]. The purpose of this article is to apply curved Casimir operators in Grassmannian geometry to construct a family of fourth order operators which are intrinsic to any Grassmannian structure and which coincide with nonstandard operators on locally flat structures. The existence of such operators was proved by a different method by Slovák and Gover in [5, theorem 5.1]. The approach via curved Casimirs gives an alternative proof of the existence and also yields new formulae for the operators, c.f. corollary 2.5. In the second part we prove by algebraic methods that these operators have the exceptional property that they are not strongly invariant.

The original inspiration comes from conformal geometry. Namely, the equivalence of Grassmannian structures and split signature conformal structures in dimension four is well known. It is obvious then that the curved version of the nonstandard operator corresponds to the conformal square of the Laplacian $\Delta^2 : \mathcal{E} \to \mathcal{E}[-4]$. Hence the construction of Δ^2 via curved Casimirs described in [3] gives us a recipe how to construct curved analogues of nonstandard operators in Grassmannian geometry. The construction is rather subtle since the direct application of the construction scheme described in [2] yields a trivial operator. This reflects the fact that Δ^2 is not strongly invariant in dimension four.

The structure of the paper is as follows. First we recall basic facts about Grassmannian geometry following from the general theory of parabolic geometries. It includes also the definitions of curved Casimir operators and nonstandard operators. The first part of the second section concerns the initial tractor bundle for the curved Casimir procedure. We derive its composition series, an explicit form of the action of one-forms on that bundle and Casimir eigenvalues on its irreducible pieces. This is used in the second part to state and prove proposition 2.4 which is the first main result of this paper. The third section is rather independent on the previous two sections. It contains the second main result, stated in proposition 3.1.

1.1 Grassmannian structures of type $(2, n)$

We shall use the conventions and the abstract index notation of [4] and [5]. An almost Grassmannian structure of type $(2, n)$ on a smooth manifold M of dimension $2n$ is given by two auxiliary vector bundles $\mathcal{E}^A$, $\mathcal{E}^{A'}$ of ranks n and 2 respectively, and identifications

$$TM = \mathcal{E}^a \cong \mathcal{E}_{A'} \otimes \mathcal{E}^A = \mathcal{E}^A_{A'}, \quad \Lambda^2 \mathcal{E}_{A'} \cong \Lambda^n \mathcal{E}^A. \tag{1}$$

Equivalently, the structure is a classical first order G-structure with reduction of the structure group $GL(2n, \mathbb{R})$ to its subgroup $S(GL(2, \mathbb{R}) \times GL(n, \mathbb{R}))$. It is well known that one can construct a unique Cartan connection associated to this structure which makes it into the |1|-graded normal parabolic geometry of type (G, P), where $G = SL(2 + n, \mathbb{R})$ and P is the stabilizer of $\mathbb{R}^2$ in $\mathbb{R}^{2+n}$. The corresponding flat model G/P are Grassmannians $\mathrm{Gr}_2(\mathbb{R}^{2+n})$. In the diagram notation, the |1|-grading of Lie algebra $\mathfrak{g} = \mathfrak{sl}(2 + n, \mathbb{R})$ defining the geometry is given by the A_{n+1}-diagram with the second node crossed. Hence $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where

$$\mathfrak{g}_0 \cong \mathfrak{s}(\mathfrak{gl}(2, \mathbb{R}) \oplus \mathfrak{gl}(n, \mathbb{R})), \quad \mathfrak{g}_1 \cong (\mathfrak{g}_{-1})^* \cong \mathbb{R}^2 \otimes \mathbb{R}^{n*}.$$

In general, the full obstruction against local flatness of a normal parabolic geometry is encoded in the harmonic curvature κ_H. In the case of the almost Grassmannian geometry of type $(2, n)$, this curvature consists of two parts: the homogeneity one component $(\kappa_H)_1$ and the homogeneity two component $(\kappa_H)_2$. For detailed description see [1, Section 4.1.3]. The former one can be interpreted geometrically as the torsion of a linear connection on the tangent bundle. The connections with such torsion $T = (\kappa_H)_1$ form a class of distinguished connections modelled on one-forms. They are also called Weyl connections and can be viewed as analogues of Levi-Civita connections in conformal geometry, see [1, section 5.1]. The harmonic curvature component $(\kappa_H)_2$ is a component of the curvature of any of these connections.

A Grassmannian geometry is usually defined as an almost Grassmannian geometry which admits a torsion free connection moreover. By [1, Theorem 4.1.1], the full obstruction against existence of such a connection is exactly the harmonic curvature component $(\kappa_H)_1$. Thus a Grassmannian geometry can be equivalently described as an almost Grassmannian geometry with $(\kappa_H)_1 = 0$ and hence it is also called semi flat in the literature. Obviously, the distinguished connections in such a case are exactly the torsion-free connections compatible with the isomorphisms in (1). In subsequent formulas for differential operators,

we will use such distinguished connections which moreover induce a flat connection on $\Lambda^2 \mathcal{E}_{A'} \cong \Lambda^n \mathcal{E}^A$. They are called closed since they form a subclass modeled on closed one forms. For more details see [1, section 5.1.7].

Let us note that there is another geometry with a similar behavior, namely the quaternionic geometry. It is given by the same grading as the Grassmannian geometry but on the quaternionic real form of $\mathfrak{g}^{\mathbb{C}}$ rather than the split form. Nevertheless, considering complexifications and complex bundles, the identifications (1) are satisfied. Thus we may include this geometry into our framework and all the constructions and results hold also for quaternionic geometry.

1.2 Curvature of distinguished connections

Theorem 4.1.1 in [1] also states that for any normal |1|-graded parabolic geometry the equation $(\kappa_H)_1 = 0$ implies that the harmonic curvature component $(\kappa_H)_2$ coincides with the so-called Weyl curvature $W := R + \partial(\mathsf{P})$. Here, R is the usual curvature tensor while ∂ is the bundle map induced by the Lie algebra differential and $\mathsf{P} \in \Omega^1(M, T^*M)$ is a uniquely defined piece of curvature called Rho-tensor. Hence according to the description of the harmonic curvature in Grassmannian geometry, the Weyl curvature is given by $W_{ab}{}^d{}_c = W_{A\ B\ C}^{A'B'D} \delta_{D'}^{C'}$, where

$$W_{A\ B\ C}^{A'B'D} \in \Gamma(\mathcal{E}^{[A'B']} \otimes (\mathcal{E}_{(ABC)}^{\ \ \ D})_0), \tag{2}$$

and where $()_0$ denotes the trace free part. See e.g. [1, section 4.1.3]. Thus the whole curvature $R_{ab}{}^d{}_c$ of a distinguished connection ∇_a is given by the sum of such $W_{ab}{}^d{}_c$ and terms which are linear and zero order in P_{ab}. The former part is obviously invariant while the latter depend on the choice of ∇_a. Namely, if the change from ∇_a to another distinguished connection $\hat{\nabla}_a$ is described by one form Υ_a, then the linearized change of the Rho-tensor is given by $\hat{\mathsf{P}}_{ab} = \mathsf{P}_{ab} + \nabla_a \Upsilon_b$. See e.g. [5] or section 5.1.8 of [1]. Moreover, it follows from the algebraic Bianchi identity that the Rho-tensor corresponding to a closed distinguished connection is symmetric, see equation (9) in [5]. The next consequence of theorem 4.1.1 in [1] is that the tensor $d^{\nabla}\mathsf{P}$, called the Cotton-York tensor, can be expressed in terms of W. The exact formula can be deduced from the differential Bianchi identity, see [9, Lemma A.5]. In particular, it follows that

$$(d^{\nabla}\mathsf{P})_{A\ B\ C}^{A'B'C'} \in \Gamma(\mathcal{E}^{[A'B']C'} \otimes \mathcal{E}_{(ABC)}). \tag{3}$$

1.3 Grassmannian bundles

Following [5] and [4], we adopt the convention $\mathcal{E}[-1] := \Lambda^2 \mathcal{E}^{A'} \cong \Lambda^n \mathcal{E}_A$, and then for $w \in \mathbb{Z}_-$ we put $\mathcal{E}[w] := \mathcal{E}[-1]^{-w}$ and $\mathcal{E}[-w] := \mathcal{E}[w]^*$. In analogy with conformal geometry, the bundle $\mathcal{E}[w]$ will be called the bundle of densities of weight w, and adding $[w]$ to the notation for a bundle indicates a tensor product by $\mathcal{E}[w]$. Motivated by the case $n = 2$ when the structure is equivalent to the spin conformal structure, the basic bundles $\mathcal{E}^A$ and $\mathcal{E}_{A'}$ are called spinor bundles, and the corresponding indices are called spinor indices. They can be raised and lowered similarly as the abstract indices in conformal or Riemannian geometry. However, we use a skew symmetric object instead of a metric which is symmetric, and thus the order of indices is important. Namely, to raise primed indices, we will use the canonical section $\epsilon^{A'B'}$ of $\mathcal{E}^{[A'B']}[1]$ which gives the isomorphism $\mathcal{E}[-1] \ni f \mapsto f\epsilon^{A'B'} \in \mathcal{E}^{[A'B']}$, and we use it in such a way that

$v^{B'} = v_{A'} \epsilon^{A'B'}$. The inverse $\epsilon_{A'B'}$ is used to lower primed indices as follows $v_{B'} = v^{A'} \epsilon_{A'B'}$.

The Grassmannian standard tractor bundle will be denoted by $\mathcal{E}^{\alpha}$. It is a vector bundle of rank $n+2$ induced by the standard representation of $\mathfrak{sl}(2+n)$ on $\mathbb{R}^{2+n}$. The obvious filtration $\mathbb{R}^{2+n} \supset \mathbb{R}^2$ gives rise to the filtration $\mathcal{E}^{\alpha} \supset \mathcal{E}^{A'}$, where $\mathcal{E}^{\alpha}/\mathcal{E}^{A'} \cong \mathcal{E}^A$. Similarly, the standard cotractor bundle $\mathcal{E}_{\alpha}$ is endowed with filtration $\mathcal{E}_{\alpha} \supset \mathcal{E}_A$, where $\mathcal{E}_{\alpha}/\mathcal{E}_A \cong \mathcal{E}_{A'}$. These data define a composition series for $\mathcal{E}_{\alpha}$ that will be denoted by $\mathcal{E}_{\alpha} = \mathcal{E}_{A'} \,\dotplus\, \mathcal{E}_A$ henceforth. This notation is motivated by the fact that summands include while there is a projection onto direct summands. Under a choice of a torsion-free connection ∇_a, the composition series splits. The exact formulas for transformations of the splittings of tractor bundles under a change from ∇_a to another torsion-free connection $\hat{\nabla}_a$ can be found by making explicit the general formulas in [1, Section 5.1], see e.g. [5]. However, we will not need these formulas in this article since we will always deal with objects and operations which are known to be invariant. The splittings will be used only to compute explicit formulas for these objects and operations.

Following [3] and [4], sections of tractor bundles will be written either as a formula in injectors $X_{\alpha}^A \in \mathcal{E}_{\alpha}^A$, $Y_{\alpha}^{A'} \in \mathcal{E}_{\alpha}^{A'}$ or simply denoted by a "vector". For example, a section v_{α} of the standard cotractor bundle reads as

$$v_{\alpha} = v_{A'} Y_{\alpha}^{A'} + v_A X_{\alpha}^A = \begin{pmatrix} v_{A'} \\ v_A \end{pmatrix}, \qquad (4)$$

where $v_{A'}$ and X_{α}^A are invariant while v_A and $Y_{\alpha}^{A'}$ depend on a choice of splitting.

1.4 Notation for forms and irreducible bundles

To simplify subsequent expressions let us adopt the index notation for forms of [4]. The usual notation $\mathcal{E}_{[AB\ldots C]}$ for kth skew symmetric power of $\mathcal{E}_A$ will be abbreviated by the following multi-indices

$$\mathbf{A}^k := [A^1 \cdots A^k], \quad k \geq 0,$$
$$\dot{\mathbf{A}}^k := [A^2 \cdots A^k], \quad k \geq 1,$$
$$\ddot{\mathbf{A}}^k := [A^3 \cdots A^k], \quad k \geq 2.$$

The subscript k will be usually omitted, i.e. $\mathcal{E}_{\mathbf{A}}$ means $\mathcal{E}_{\mathbf{A}^k}$ throughout the article. Also the square bracket may be absent. Indices labelled with sequential superscripts automatically indicate a completely skew set of indices. The same notation will be used also for skew symmetric powers of tractor bundles. For example, the bundle $\mathcal{E}_{\alpha^2\ldots\alpha^k} = \mathcal{E}_{[\alpha^2\ldots\alpha^k]}$ will be denoted by $\mathcal{E}_{\dot{\boldsymbol{\alpha}}}$. We will combine this notation with the usual Young diagram notation for irreducible bundles induced by $SL(n)$-modules as follows. We adopt the convention that we symmetrized over sets of indices corresponding to the rows of the diagram first and then with the result skew over sets of indices corresponding to the columns of the diagram. The sets of indices in columns may be then shortened by multi-indices as above. For instance suppose we have a general valence four spinor $A_{A^1 A^2 B^1 B^2} \in \mathcal{E}_{A^1 A^2 B^1 B^2}[w]$. If we first symmetrized as follows

$B_{A^1 A^2 B^1 B^2} := A_{(A^1|A^2|B^1)B^2}$, $C_{A^1 A^2 B^1 B^2} := B_{A^1(A^2|B^1|B^2)}$ and then on this result skew over the nonsymmetric indices $D_{\mathbf{A}^2 \mathbf{B}^2} := C_{[A^1 A^2][B^1 B^2]}$, then

$$D_{\mathbf{A}^2 \mathbf{B}^2} \in \boxplus \mathcal{E}_{\mathbf{A}^2 \mathbf{B}^2}[w].$$

1.5 Nonstandard operators

Using the above notation, we can explicitly write down irreducible subbundles of k-forms. Namely, since $T^*M = \mathcal{E}^{A'} \otimes \mathcal{E}_A$ by (1), each such subbundle is given by a product of an irreducible $SL(2)$-bundle in $\otimes^k \mathcal{E}^{A'}$, which is a symmetric power of $\mathcal{E}^{A'}$, and an irreducible $SL(n)$-bundle in $\otimes^k \mathcal{E}_A$ indicated by the corresponding diagram. For example, the bundle of two-forms splits into $\mathcal{E}^{(A'B')} \otimes \mathcal{E}_{\mathbf{A}^2}$ and $\mathcal{E}_{(AB)}[-1]$, while the decomposition of $\Lambda^4 T^*M$ reads as

$$\Lambda^4(\mathcal{E}_A^{A'}) = \boxplus \mathcal{E}_{\mathbf{A}^2 \mathbf{B}^2}[-2] \oplus \mathcal{E}^{(A'B')} \otimes \boxplus \mathcal{E}_{\mathbf{A}^3 \mathbf{B}^1}[-1] \oplus \mathcal{E}_{\mathbf{A}^4}^{(A'B'C'D')}.$$

The decomposition of $\Lambda^k T^*M$ can be written a similar form for any k. In particular, the number of columns of Young diagrams that occur is never more than two since this would correspond to a k-form which would be skew symmetric in more than two primed spinor indices, and such forms vanish due to the low rank of $\mathcal{E}^{A'}$. For instance, the irreducible bundles of $2k$-forms are bundles which are given for each $0 \le \ell \le k$ by the tensor product of $S^{2\ell} \mathcal{E}^{A'}[\ell - k]$ with the subbundle of $\otimes^{2k} \mathcal{E}_A$ which is given by the Young diagram with columns of height $k + \ell$ and $k - \ell$. Of course, if $k + \ell > n$ then the component vanishes, and if $k + \ell = n$ then we get a one more copy of $\mathcal{E}[-1] \cong \Lambda^n \mathcal{E}_A$ and the Young diagram contains n boxes less. In particular, we see that for $2k \le n$, the bundle of $2k$-forms decomposes into $k+1$ components, while for $2k > n$ we have $n - k + 1$ components. The decomposition of odd degree forms is similar, and so the De Rham sequence splits into a triangular pattern, see [11] for more details.

We mainly are interested in the long side of this triangle. In the case of a locally flat Grassmannian structure, there occur invariant operators which correspond to nonstandard homomorphisms of generalized Verma modules, cf. [6] and [7]. Concretely, for each $2 \le k \le n$ there is a fourth order invariant operator

$$\widetilde{\Box}_{ABCD} : \overset{k-2}{\updownarrow} \boxplus \mathcal{E}_{\ddot{\mathbf{A}}\ddot{\mathbf{B}}}[-k+2] \to \overset{k}{\updownarrow} \boxplus \mathcal{E}_{\mathbf{A}\mathbf{B}}[-k].$$

These operators are called the nonstandard operators for Grassmannian structures. In terms of preferred flat connections (which are known to exist on locally flat structures), the operator $\widetilde{\Box}_{ABCD}$ is obviously given by $\nabla_{I'A} \nabla_B^{I'} \nabla_{J'C} \nabla_D^{J'}$, followed by the projection to the target bundle. Let us note that such projection is unique since the commutativity of $\nabla_A^{A'}$ ensures that $\nabla_{I'A} \nabla_B^{I'} \nabla_{J'C} \nabla_D^{J'}$ is a section of $\mathcal{E}_{[AB][CD]}[-2]$, and the target bundle appears only in the tensor product of the initial bundle with $\boxplus \mathcal{E}_{ABCD}[-2]$ which occurs with multiplicity one in $\mathcal{E}_{[AB][CD]}[-2]$.

All other invariant operators between exterior forms are components of exterior derivative and their nonzero compositions (of order two). Therefore, they exist also on general almost Grassmannian structures. On the other hand, it may be proved that the nonstandard operators are not strongly invariant, see [8]. Hence it is not clear whether these extend to invariant operators to a larger class of (almost) Grassmannian structures. However, it was proved by J. Slovák

and R. Gover in [5, Theorem 5.1] that they all do exist on any Grassmannian structure of type $(2, n)$. In sequel, we construct them via so called curved Casimir operators, and thus we give an alternative proof of their existence and alternative formulae for them.

1.6 A formula for the curved Casimir operator, construction principle

First recall from Theorem 3.4 of [2] that the curved Casimir $\mathcal{C}$ is and invariant operator which on an irreducible bundle $W \to M$ acts by a real multiple of the identity. We denote the corresponding scalar by β_W and we call it the Casimir eigenvalue. By the theorem, this scalar can be computed in terms of weights of the representation which induces W. Namely, if the lowest weight of this representation is $-\lambda$, then

$$\beta_W = \langle \lambda, \lambda + 2\rho \rangle, \tag{5}$$

where ρ is the lowest form which by definition equals half the sum of all positive roots or equivalently, it equals the sum of all fundamental weights.

A suitable formula for $\mathcal{C}$ on an arbitrary natural Grassmannian bundle $V \to M$ can be obtained from the formula in terms of an adapted local frame for the adjoint tractor bundle from Proposition 3.3 of [2]. Following the proof of Proposition 2.2 in [3], which gives a formula for $\mathcal{C}$ in conformal geometry, one gets a formula which has precisely the same form as the formula in this proposition. Namely, having fixed a connection ∇ from the class of the distinguished connections, the adjoint tractor bundle splits as $TM \oplus \mathrm{End}_0(TM) \oplus T^*M$, and having chosen a local orthonormal frame ξ_ℓ for TM with dual frame φ^ℓ for T^*M (with respect to the Cartan-Killing form), the formula for the curved Casimir operator reads as

$$\mathcal{C}(s) = \beta(s) - 2\sum_\ell \varphi^\ell \bullet \nabla_{\psi_\ell} s - 2 \sum_\ell \varphi^\ell \bullet \mathsf{P}(\psi_\ell) \bullet s, \tag{6}$$

where $\beta : V \to V$ is the bundle map which acts on each irreducible component $W \subset V$ by multiplication by β_W. For the sake of simplicity, we write this formula in a shortened form as $\mathcal{C} = \beta_W - 2\nabla \bullet -2\mathsf{P} \bullet \bullet$.

The construction principle we use is inspired by the construction of splitting operators in section 3.5 of [2]. Let $\mathcal{T} \to M$ be a tractor bundle. The filtration of the standard tractor bundle induces a natural filtration $\mathcal{T} = \mathcal{T}^0 \supset \mathcal{T}^1 \supset \cdots \supset \mathcal{T}^N$, which we write as $\mathcal{T} = \mathcal{T}^0/\mathcal{T}^1 \oplus \mathcal{T}^1/\mathcal{T}^2 \oplus \cdots \oplus \mathcal{T}^N$. Each of the subquotients $\mathcal{T}^i/\mathcal{T}^{i+1}$ splits into a direct sum of irreducible tensor bundles. We know from above that on sections of each such irreducible $W \subset \mathcal{T}^i/\mathcal{T}^{i+1}$, the curved Casimir acts by multiplication with β_W. Let $\beta_j^1, \ldots, \beta_j^{n_j}$ denote for all $j > i$ the different eigenvalues that occur in the decomposition of $\mathcal{T}^j/\mathcal{T}^{j+1}$. Then the natural operator on $\Gamma(VM)$ defined by $L := \Pi_{j=i+1}^N \Pi_{\ell=1}^{n_j}(\mathcal{C} - \beta_j^\ell)$ descends to an operator $\Gamma(WM) \to \Gamma(V^iM)$. Moreover, if for $j > i$ all eigenvalues β_j^ℓ are different from β_W, then L defines a natural splitting operator. On the other hand, if an eigenvalue β_j^ℓ corresponding to an irreducible bundle $W'M \subset V^jM/V^{j+1}M$ coincides with β_W, then L restricted to WM defines an invariant operator $\Gamma(WM) \to \Gamma(W'M)$. A detailed explanation of the construction principle can be found in section 2.3 of [3].

2 Curved Casimir construction

The first step of the construction of nonstandard operators is a choice of a suitable tractor bundle. Following the construction of Paneitz operator in [3], it is natural to take as the input bundle such a tractor bundle, which has the initial bundle and the target bundle of the operator in the top slot and the bottom slot respectively. Hence from the description of $\widetilde{\square}_{ABCD}$ in section 1.5 we conclude that the right input tractor bundle for our construction is

$$\mathcal{T} := \overset{\uparrow}{\underset{\downarrow}{k}} \; \boxplus \mathcal{E}_{\boldsymbol{\alpha\beta}}[-k]. \tag{7}$$

2.1 The composition series of $\mathcal{T}$

The composition series for this tractor bundle can be computed from the series of the simpler bundles. Namely, it follows from the structure of the standard cotractor bundle (4) that $\mathcal{E}_{[\alpha\beta]} = \mathcal{E}_{[A'B']} \oplus \mathcal{E}_{A'B} \oplus \mathcal{E}_{[AB]}$. In terms of densities and multi indices, defined in 1.3 and 1.4 respectively, this can be also written as $\mathcal{E}_{\boldsymbol{\alpha}^2} = \mathcal{E}[1] \oplus \mathcal{E}_A^{A'}[1] \oplus \mathcal{E}_{\mathbf{A}^2}$. It is then easy to see that for an arbitrary k the composition series of k-forms is

$$\mathcal{E}_{\boldsymbol{\alpha}} = \mathcal{E}_{\ddot{\mathbf{A}}}[1] \oplus \mathcal{E}_{\dot{\mathbf{A}}}^{A'}[1] \oplus \mathcal{E}_{\mathbf{A}}.$$

From this composition series, one easily deduces that the second symmetric power satisfies

$$\mathcal{E}_{(\boldsymbol{\alpha\beta})} = \mathcal{E}_{(\ddot{\mathbf{A}}\ddot{\mathbf{B}})}[2] \oplus \mathcal{E}_{\dot{\mathbf{A}}\ddot{\mathbf{B}}}^{A'}[2] \oplus (\mathcal{E}_{(\dot{\mathbf{A}}\dot{\mathbf{B}})}^{(A'B')}[2] \oplus \mathcal{E}_{[\dot{\mathbf{A}}\dot{\mathbf{B}}]}[1] \oplus \mathcal{E}_{\mathbf{A}\ddot{\mathbf{B}}}[1]) \oplus \mathcal{E}_{\mathbf{A}\dot{\mathbf{B}}}^{A'}[1] \oplus \mathcal{E}_{(\mathbf{AB})}.$$

The bundles displayed are not irreducible however. On the left-hand side, there appears $\mathcal{T}$ as a direct summand. It is an easy observation that the diagrams corresponding to the other summands have also two columns at most which means that the height of one of them is greater then k. Hence the desired composition series of $\mathcal{T}$ can be detected in the right-hand side of the previous equation by considering only such terms that vanish under alternations over more than k tractor indices. Its exact form is given in the lemma below.

In order to write down the representatives of the the bundles in the composition series explicitly, it is convenient to use a notation analogous to the notation of [4]. Concretely, we use the injectors X_α^A, $Y_\alpha^{A'}$ from (4) to define new injectors

$$\begin{aligned}
\mathbb{X}_{\boldsymbol{\alpha}}^{\mathbf{A}} &:= X_{[\alpha_1}^{A_1} \ldots X_{\alpha_k]}^{A_k} \in \mathcal{E}_{\boldsymbol{\alpha}}^{\mathbf{A}}, \\
\mathbb{W}_{\boldsymbol{\alpha}}^{A'\dot{\mathbf{A}}} &:= Y_{[\alpha_1}^{A'} X_{\alpha_2}^{A_2} \ldots X_{\alpha_k]}^{A_k} \in \mathcal{E}_{\boldsymbol{\alpha}}^{A'\dot{\mathbf{A}}}, \\
\mathbb{Y}_{\boldsymbol{\alpha}}^{A'B'\ddot{\mathbf{A}}} &:= Y_{[\alpha_1}^{A'} Y_{\alpha_2}^{B'} X_{\alpha_3}^{A_3} \ldots X_{\alpha_k]}^{A_k} \in \mathcal{E}_{\boldsymbol{\alpha}}^{[A'B']\ddot{\mathbf{A}}}.
\end{aligned} \tag{8}$$

In terms of these injectors, a tractor k-form $\varphi_{\boldsymbol{\alpha}}$ can be simply expressed as

$$\varphi_{\boldsymbol{\alpha}} = \sigma_{\ddot{\mathbf{A}}} \epsilon_{A'B'} \mathbb{Y}_{\boldsymbol{\alpha}}^{A'B'\ddot{\mathbf{A}}} + \mu_{\dot{\mathbf{A}}}^{A'} \epsilon_{A'B'} \mathbb{W}_{\boldsymbol{\alpha}}^{B'\dot{\mathbf{A}}} + \rho_{\mathbf{A}} \mathbb{X}_{\boldsymbol{\alpha}}^{\mathbf{A}},$$

where $\sigma_{\ddot{\mathbf{A}}} \in \mathcal{E}_{\ddot{\mathbf{A}}}[1]$, $\mu_{\dot{\mathbf{A}}}^{A'} \in \mathcal{E}_{\dot{\mathbf{A}}}^{A'}[1]$, $\rho_{\mathbf{A}} \in \mathcal{E}_{\mathbf{A}}$. In a similar way, one can explicitly write down the sections of $\mathcal{E}_{(\boldsymbol{\alpha\beta})}$ and also the sections of its subbundles, in particular the sections of our bundle $\mathcal{T}$.

Figure 1: Composition series of $\mathcal{T}$

$$
\mathcal{T} := \boxed{}\,\mathcal{E}_{\boldsymbol{\alpha\beta}}[-k] =
\begin{pmatrix}
\boxed{}\,\mathcal{E}_{\ddot{\mathbf{A}}\ddot{\mathbf{B}}}[-k+2] \\[4pt]
\oplus \\[4pt]
\mathcal{E}^{A'}\otimes\boxed{}\,\mathcal{E}_{\dot{\mathbf{A}}\ddot{\mathbf{B}}}[-k+2] \\[4pt]
\oplus \\[4pt]
\mathcal{E}^{(A'B')}\otimes\boxed{}\,\mathcal{E}_{\dot{\mathbf{A}}\dot{\mathbf{B}}}[-k+2]\ \oplus\ \boxed{}\,\mathcal{E}_{\mathbf{A}\ddot{\mathbf{B}}}[-k+1] \\[4pt]
\oplus \\[4pt]
\mathcal{E}^{A'}\otimes\boxed{}\,\mathcal{E}_{\mathbf{A}\dot{\mathbf{B}}}[-k+1] \\[4pt]
\oplus \\[4pt]
\boxed{}\,\mathcal{E}_{\mathbf{A}\mathbf{B}}[-k]
\end{pmatrix}
\tag{9}
$$

Lemma 2.1 *The composition series of tractor bundle $\mathcal{T}$ from (7) has the form as displayed in figure 2.1. In terms of injectors defined by (8), its section $v_{\boldsymbol{\alpha\beta}}$ can be expressed as*

$$
\begin{aligned}
v_{\boldsymbol{\alpha\beta}} =\ & \sigma_{\ddot{\mathbf{A}}\ddot{\mathbf{B}}}\epsilon_{A'B'}\epsilon_{C'D'}\mathbb{Y}^{A'B'\ddot{\mathbf{A}}}_{(\boldsymbol{\alpha}}\mathbb{Y}^{C'D'\ddot{\mathbf{B}}}_{\boldsymbol{\beta})} + \mu^{A'}_{\dot{\mathbf{A}}\ddot{\mathbf{B}}}\epsilon_{A'B'}\epsilon_{C'D'}\mathbb{W}^{B'\dot{\mathbf{A}}}_{(\boldsymbol{\alpha}}\mathbb{Y}^{C'D'\ddot{\mathbf{B}}}_{\boldsymbol{\beta})} \\
&+ A^{A'B'}_{\dot{\mathbf{A}}\dot{\mathbf{B}}}\epsilon_{A'C'}\epsilon_{B'D'}\mathbb{W}^{C'\dot{\mathbf{A}}}_{(\boldsymbol{\alpha}}\mathbb{W}^{D'\dot{\mathbf{B}}}_{\boldsymbol{\beta})} + \alpha_{\mathbf{A}\ddot{\mathbf{B}}}\epsilon_{C'D'}(\mathbb{X}^{\mathbf{A}}_{(\boldsymbol{\alpha}}\mathbb{Y}^{C'D'\ddot{\mathbf{B}}}_{\boldsymbol{\beta})} + \tfrac{k}{2}\mathbb{W}^{C'A_1\ddot{\mathbf{B}}}_{(\boldsymbol{\alpha}}\mathbb{W}^{D'\dot{\mathbf{A}}}_{\boldsymbol{\beta})}) \\
&+ \nu^{A'}_{\mathbf{A}\dot{\mathbf{B}}}\epsilon_{A'B'}\mathbb{X}^{\mathbf{A}}_{(\boldsymbol{\alpha}}\mathbb{W}^{B'\dot{\mathbf{B}}}_{\boldsymbol{\beta})} + \rho_{\mathbf{A}\mathbf{B}}\mathbb{X}^{\mathbf{A}}_{(\boldsymbol{\alpha}}\mathbb{X}^{\mathbf{B}}_{\boldsymbol{\beta})},
\end{aligned}
\tag{10}
$$

where $\sigma_{\ddot{\mathbf{A}}\ddot{\mathbf{B}}}$, $\mu^{A'}_{\dot{\mathbf{A}}\ddot{\mathbf{B}}}$, $A^{A'B'}_{\dot{\mathbf{A}}\dot{\mathbf{B}}}$, $\alpha_{\mathbf{A}\ddot{\mathbf{B}}}$, $\nu^{A'}_{\mathbf{A}\dot{\mathbf{B}}}$, $\rho_{\mathbf{A}\mathbf{B}}$ are representatives of the corresponding bundles appearing in the composition series of $\mathcal{T}$.

By definition, sections of our bundle are such sections of $\mathcal{E}_{(\boldsymbol{\alpha\beta})}[-k]$ that vanish under all alternations over more than k indices. It is an easy observation that the low dimension of $\mathcal{E}^{A'}$ implies that this condition translates from tractor indices to unprimed spinor indices. Hence the height of Young diagrams of all bundles in the composition series of our bundle is less or equal to k. Looking at the composition series of $\mathcal{E}_{(\boldsymbol{\alpha\beta})}$, this condition determines a unique irreducible bundle in each summand. Twisting everything by weight $-k$, these bundles already are the bundles appearing in (9). Their representatives in our tractor bundle are obvious except for the bundle with Young diagram with columns of heights k and $k-2$. One copy of this bundle sits in $\mathcal{E}_{\mathbf{A}\dot{\mathbf{B}}}[-k+1]$ and the second in $\mathcal{E}_{\mathbf{A}\ddot{\mathbf{B}}}[-k+1]$, and the condition on tractor indices yields a nontrivial relation between these two copies. Namely, we will show that the corresponding sections $B_{\dot{\mathbf{A}}\dot{\mathbf{B}}}$ and $\alpha_{\mathbf{A}\ddot{\mathbf{B}}}$ are related by

$$
B_{\dot{\mathbf{A}}\dot{\mathbf{B}}} = \frac{k}{2}\alpha_{[A_2|\dot{\mathbf{B}}|\ddot{\mathbf{A}}]}, \quad \alpha_{\mathbf{A}\ddot{\mathbf{B}}} = (-1)^k(k-1)B_{\mathbf{A}\ddot{\mathbf{B}}}.
\tag{11}
$$

This will finish the proof since the first equation shows that a section $\alpha_{\mathbf{A}\ddot{\mathbf{B}}}$ of the

bundle with Young diagram with columns of heights k and $k-2$ is represented in our tractor bundle by

$$\alpha_{\mathbf{A}\ddot{\mathbf{B}}}\epsilon_{C'D'}(\mathbb{X}^{\mathbf{A}}_{(\boldsymbol{\alpha}}\mathbb{Y}^{C'D'\ddot{\mathbf{B}}}_{\boldsymbol{\beta})} + \frac{k}{2}\mathbb{W}^{C'A_1\ddot{\mathbf{B}}}_{(\boldsymbol{\alpha}}\mathbb{W}^{D'\dot{\mathbf{A}}}_{\boldsymbol{\beta})}).$$

Equations (11) relating these two sections can be found by looking at the middle slots of $v_{[\boldsymbol{\alpha}\beta_1]\dot{\boldsymbol{\beta}}} = 0$. Namely, expanding the alternation in $\boldsymbol{\beta}$ with respect to β_1 and $\dot{\boldsymbol{\beta}}$ yields

$$\mathbb{W}^{C'\dot{\mathbf{A}}}_{[\boldsymbol{\alpha}}\mathbb{W}^{D'\dot{\mathbf{B}}}_{\beta_1]\dot{\boldsymbol{\beta}}} = (-1)^{k-1}\frac{1}{k}\mathbb{Y}^{C'D'\dot{\mathbf{A}}}_{[\boldsymbol{\alpha}\beta_1]}\mathbb{X}^{\dot{\mathbf{B}}}_{\dot{\boldsymbol{\beta}}} + \frac{1}{k}\sum_{i=2}^{k}(-1)^{i-1}\mathbb{W}^{C'\dot{\mathbf{A}}B_i}_{[\boldsymbol{\alpha}\beta_1]}\mathbb{W}^{D'B_2\cdots\hat{B}_i\cdots B_k}_{\dot{\boldsymbol{\beta}}}$$

and since $B_{\dot{\mathbf{A}}\dot{\mathbf{B}}}$ is skew symmetric in $B_2, \ldots, B_k$, the alternation over tractor indices $\alpha_1, \ldots, \alpha_k$ and β_1 of term $B_{\dot{\mathbf{A}}\dot{\mathbf{B}}}\epsilon_{C'D'}\mathbb{W}^{C'\dot{\mathbf{A}}}_{(\boldsymbol{\alpha}}\mathbb{W}^{D'\dot{\mathbf{B}}}_{\boldsymbol{\beta})}$ is equal to

$$B_{\dot{\mathbf{A}}\dot{\mathbf{B}}}\epsilon_{C'D'}\left((-1)^{k-1}\frac{1}{k}\mathbb{Y}^{C'D'\dot{\mathbf{A}}}_{[\boldsymbol{\alpha}\beta_1]}\mathbb{X}^{\dot{\mathbf{B}}}_{\dot{\boldsymbol{\beta}}} - \frac{k-1}{k}\mathbb{W}^{C'\dot{\mathbf{A}}B_2}_{[\boldsymbol{\alpha}\beta_1]}\mathbb{W}^{D'\ddot{\mathbf{B}}}_{\dot{\boldsymbol{\beta}}}\right), \tag{12}$$

while the alternation of term $A^{A'B'}_{\dot{\mathbf{A}}\dot{\mathbf{B}}}\epsilon_{A'C'}\epsilon_{B'D'}\mathbb{W}^{C'\dot{\mathbf{A}}}_{(\boldsymbol{\alpha}}\mathbb{W}^{D'\dot{\mathbf{B}}}_{\boldsymbol{\beta})}$ over the same tractor indices obviously vanishes. Similarly, for the other injectors the expansion of $\boldsymbol{\beta}$ with respect to β_1 and $\dot{\boldsymbol{\beta}}$ yields

$$\mathbb{X}^{\mathbf{A}}_{[\boldsymbol{\alpha}}\mathbb{Y}^{C'D'\ddot{\mathbf{B}}}_{\beta_1]\dot{\boldsymbol{\beta}}} = (-1)^k\frac{2}{k}\mathbb{W}^{[C'|\mathbf{A}|}_{[\boldsymbol{\alpha}\beta_1}\mathbb{W}^{D']\ddot{\mathbf{B}}}_{\dot{\boldsymbol{\beta}}} - \frac{1}{k}\sum_{i=3}^{k}(-1)^i\mathbb{X}^{\mathbf{A}B_i}_{[\boldsymbol{\alpha}\beta_1]}\mathbb{Y}^{C'D'B_3\cdots\hat{B}_i\cdots B_k}_{\dot{\boldsymbol{\beta}}}$$

and

$$\mathbb{X}^{\mathbf{A}}_{[\beta_1|\boldsymbol{\beta}|}\mathbb{Y}^{C'D'\ddot{\mathbf{B}}}_{\boldsymbol{\alpha}]} = \frac{1}{k}\sum_{i=1}^{k}(-1)^{i-1}\mathbb{Y}^{C'D'\ddot{\mathbf{B}}A_i}_{[\boldsymbol{\alpha}\beta_1]}\mathbb{X}^{A_1\cdots\hat{A}_i\cdots A_k}_{\dot{\boldsymbol{\beta}}}.$$

Concerning the first formula, the second term on the right vanishes when applied to $\alpha_{\mathbf{A}\ddot{\mathbf{B}}}$ since each summand is skew symmetric in $k+1$ unprimed indices. Skewing the second formula in $A_1, \ldots, A_k$ one obtains that the alternation over tractor indices $\alpha_1, \ldots, \alpha_k$ and β_1 of term $\alpha_{\mathbf{A}\ddot{\mathbf{B}}}\epsilon_{C'D'}\mathbb{X}^{\mathbf{A}}_{(\boldsymbol{\alpha}}\mathbb{Y}^{C'D'\ddot{\mathbf{B}}}_{\boldsymbol{\beta})}$ is

$$\alpha_{\mathbf{A}\ddot{\mathbf{B}}}\epsilon_{C'D'}\left((-1)^k\frac{1}{k}(\mathbb{W}^{C'\mathbf{A}}_{[\boldsymbol{\alpha}\beta_1]}\mathbb{W}^{D'\ddot{\mathbf{B}}}_{\dot{\boldsymbol{\beta}}} + \frac{1}{2}\mathbb{Y}^{C'D'\ddot{\mathbf{B}}A_1}_{[\boldsymbol{\alpha}\beta_1]}\mathbb{X}^{\dot{\mathbf{A}}}_{\dot{\boldsymbol{\beta}}}\right). \tag{13}$$

Then $v_{[\boldsymbol{\alpha}\beta_1]\dot{\boldsymbol{\beta}}} = 0$ implies that the sum of (12) and (13) is equal to zero, and this immediately gives equations (11).

2.2 The action of one forms on $\mathcal{T}$

The dual to standard representation of $\mathfrak{sl}(2+n)$ on $\mathbb{R}^{2+n}$ restricted to $\mathfrak{g}_1$ translates to bundles and gives rise to an action of $\mathcal{E}_a$ on $\mathcal{E}_\alpha$. It is easy to see that for an one-form $\varphi^{A'}_A \in \mathcal{E}^{A'}_A = \mathcal{E}_a$ and a standard cotractor $v_\alpha = v_{A'}Y^{A'}_\alpha + v_a X^A_\alpha$ its explicit form reads as $\varphi \bullet v_\alpha = -\varphi^{A'}_A v_{A'}X^A_\alpha$. Hence the action of $\varphi^{A'}_A$ on the injectors is given by equations

$$(\varphi \bullet Y)^{A'}_\alpha = -\varphi^{A'}_A X^A_\alpha, \quad \varphi \bullet X = 0.$$

Using these basic relations one is able to compute easily the action of one forms on more complicated tractor bundles.

Lemma 2.2 *The action of an one form $\varphi_A^{A'} \in \mathcal{E}_A^{A'}$ on a section of $\mathcal{T}$ is given by*

$$
\varphi_A^{A'} \bullet \begin{pmatrix} \sigma_{\ddot{\mathbf{A}}\ddot{\mathbf{B}}} \\ \mu_{\dot{\mathbf{A}}\ddot{\mathbf{B}}}^{A'} \\ A_{\dot{\mathbf{A}}\dot{\mathbf{B}}}^{A'B'}|\alpha_{\mathbf{A}\ddot{\mathbf{B}}} \\ \nu_{\mathbf{A}\dot{\mathbf{B}}}^{A'} \\ \rho_{\mathbf{A}\mathbf{B}} \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi_{[A_2}^{A'}\sigma_{\ddot{\mathbf{A}}]\ddot{\mathbf{B}}} \\ \varphi_{[A_2}^{(A'}\mu_{|\dot{\mathbf{B}}|\ddot{\mathbf{A}}]}^{B')} + \varphi_{[B_2}^{(A'}\mu_{|\dot{\mathbf{A}}|\ddot{\mathbf{B}}]}^{B')} \quad | \quad \varphi_{I'[A_1}\mu_{\dot{\mathbf{A}}]\ddot{\mathbf{B}}}^{I'} \\ 2\varphi_{I'[A_1}A_{\dot{\mathbf{A}}]\dot{\mathbf{B}}}^{I'A'} + 2\varphi_{[B_2}^{A'}\alpha_{|\mathbf{A}|\ddot{\mathbf{B}}]} - k(-1)^k \varphi_{[A_1}^{A'}\alpha_{\dot{\mathbf{A}}]\dot{\mathbf{B}}} \\ \varphi_{I'[A_1}\nu_{|\mathbf{B}|\dot{\mathbf{A}}]}^{I'} + \varphi_{I'[B_1}\nu_{|\mathbf{A}|\dot{\mathbf{B}}]}^{I'} \end{pmatrix}
$$
(14)

From the defining equations (8) for the injectors $\mathbb{X}_{\boldsymbol{\alpha}}^{\mathbf{A}}$, $\mathbb{W}_{\boldsymbol{\alpha}}^{A'\dot{\mathbf{A}}}$, $\mathbb{Y}_{\boldsymbol{\alpha}}^{A'B'\ddot{\mathbf{A}}}$ and from the basic relations for the action of an one form $\varphi_A^{A'} \in \mathcal{E}_A^{A'}$ on injectors $X_\alpha^A, Y_\alpha^{A'}$, one immediately derives

$$
(\varphi \bullet \mathbb{Y})_{\boldsymbol{\alpha}}^{A'B'\ddot{\mathbf{A}}} = 2\varphi_I^{[A'}\mathbb{W}^{B']I\ddot{\mathbf{A}}}
$$

$$
(\varphi \bullet \mathbb{W})_{\boldsymbol{\alpha}}^{A'\dot{\mathbf{A}}} = -\varphi_I^{A'}\mathbb{X}_{\boldsymbol{\alpha}}^{I\dot{\mathbf{A}}}
$$

$$
\varphi \bullet \mathbb{X} = 0.
$$

Now a direct use of these equations to (10) yields the result.

2.3 Casimir eigenvalues on $\mathcal{T}$

Knowing the irreducible bundles occurring in the composition series of $\mathcal{T}$, we can directly compute the corresponding Casimir eigenvalues using formula (5). Since the minus lowest weights of basic bundles $\mathcal{E}^{A'}$, $\mathcal{E}[1]$ and $\Lambda^k \mathcal{E}_A = \mathcal{E}_{\mathbf{A}}$ are given by $\omega_1 - \omega_2$, ω_2 and $\omega_{k+2} - \omega_2$ respectively, where ω_k for $k = 1, \ldots, n-1$ denotes fundamental weights of $\mathfrak{g}$, the minus lowest weights corresponding to the bundles in (9) are

$$
\begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2^1 \quad | \quad \lambda_2^2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 2\omega_k - k\omega_2 \\ \omega_{k+1} + \omega_k - (k+1)\omega_2 + \omega_1 \\ 2\omega_{k+1} - (k+2)\omega_2 + 2\omega_1 \quad | \quad \omega_{k+2} + \omega_k - (k+1)\omega_2 \\ \omega_{k+2} + \omega_{k+1} - (k+2)\omega_2 + \omega_1 \\ 2\omega_{k+2} - (k+2)\omega_2 \end{pmatrix}.
$$

The computation of Casimir eigenvalues from formula (5) is parallel to [3] using these weights as an input. We get the following.

Lemma 2.3 *The eigenvalues of the curved Casimir operator on the irreducible*

bundles appearing in the composition series of $\mathcal{T}$ read as

$$
\begin{pmatrix}
& \beta_0 & \\
& \beta_1 & \\
\beta_2^1 & | & \beta_2^2 \\
& \beta_3 & \\
& \beta_4 &
\end{pmatrix}
=
\begin{pmatrix}
& 0 & \\
& 0 & \\
4 & | & -4 \\
& 0 & \\
& 0 &
\end{pmatrix}.
$$

2.4 Construction of nonstandard operators

Looking at Casimir eigenvalues for irreducible components of the tractor bundle $\mathcal{T}$ displayed in lemma 2.3 we see that the candidates for curved analogues of the nonstandard operators $\widetilde{\square}_{ABCD}$ are induced by $\mathcal{C}^3 \circ (\mathcal{C} - 4) \circ (\mathcal{C} + 4)$. By the construction principle, such a composition of the curved Casimirs gives an invariant operator between the top slot and the bottom slot which are exactly the bundles where the nonstandard operators is defined. However, we will show that such direct construction yields always a trivial operator. The reason for this might be seen in the high degeneracy of this case – there appear four zeros among the Casimir eigenvalues. On the other hand, the curved Casimirs induce more operators due to the degeneracy. Namely, the operator $\mathcal{C}$ itself evidently gives an invariant operator between first two slots and an invariant operator between the last two slots. It is easy to see that these operators are nothing else but the exterior derivatives. The next operator we get is due to the coincidence of eigenvalues $\beta_1 = \beta_3$. By the construction principle, an invariant operator between the respective bundles

$$
\mathsf{M}_{AB} : \mathcal{E}^{A'} \otimes \boxed{\vdots} \mathcal{E}_{\mathbf{\dot{A}\ddot{B}}}[-k+2] \to \mathcal{E}^{A'} \otimes \boxed{\vdots} \mathcal{E}_{\mathbf{A\dot{B}}}[-k+1]
$$

is induced by $\mathcal{C} \circ (\mathcal{C} - 4) \circ (\mathcal{C} + 4)$. Similarly, the coincidences $\beta_0 = \beta_3$ and $\beta_1 = \beta_4$ show that $\mathcal{C}^2 \circ (\mathcal{C} - 4) \circ (\mathcal{C} + 4)$ gives rise to invariant operators between the corresponding bundles. Obviously, these operators are given by the compositions $\mathsf{M} \circ d$ and $d \circ \mathsf{M}$. But, there are no such operators in the classification of invariant operators on flat structures, and so the compositions must vanish in that case. We will prove that $\mathsf{M} \circ d$ and $d \circ \mathsf{M}$ vanish identically even in the case of a general torsion-free structure with a nonzero curvature. It implies then that $\mathcal{C}^2 \circ (\mathcal{C} - 4) \circ (\mathcal{C} + 4)$ actually induces an invariant operator from the top slot to the bottom slot, and we will show that this is the curved analogue of the nonstandard operator that we wanted to construct. Let us remark that this construction breaks down in the case that the torsion does not vanish, see Remark 2.6 bellow.

Proposition 2.4 *Let M be a manifold endowed with Grassmannian or quaternionic structure. For each integer k such that $2 \le k \le n$ the action of operator $\mathcal{C}^2 \circ (\mathcal{C} - 4) \circ (\mathcal{C} + 4)$ on tractor bundle $\mathcal{T}$ gives rise to a fourth order invariant operator*

$$
\square_{ABCD} : {\overset{k-2}{\updownarrow}} \boxed{} \mathcal{E}_{\mathbf{\ddot{A}\ddot{B}}}[-k+2] \to {\overset{k}{\updownarrow}} \boxed{} \mathcal{E}_{\mathbf{AB}}[-k]
$$

which coincides with the corresponding nonstandard operator on flat structures.

First, we prove that for each k such that $0 \leq k \leq n-2$ the operators $\mathsf{M} \circ d$ and $d \circ \mathsf{M}$ vanish identically. In the second step, we prove that the principal part of $\Box_{ABCD}$ is given by a nonzero scalar multiple of the corresponding nonstandard operator $\tilde{\Box}_{ABCD}$, and thus the operators coincide on the category of locally flat structures.

The direct way how to prove vanishing of the two operators is to express the operator M in terms of the Weyl connection and Rho-tensor. This can be easily done in a way since the operator is obtained by acting with $\mathcal{C} \circ (\mathcal{C} - 4) \circ (\mathcal{C} + 4)$, and we have formula (6) for $\mathcal{C}$ at disposal. Namely, denoting the projections to the left slot (corresponding to β_2^1) and right slot (corresponding to β_2^2) by $(\ )_1$ and $(\ )_2$ respectively, it is an easy exercise to show that up to a scalar multiple M can be written as

$$\mathsf{M} = \nabla \bullet (\nabla \bullet \ )_1 - \nabla \bullet (\nabla \bullet \ )_2 + 2\mathsf{P} \bullet \bullet \ . \qquad (15)$$

For more details see [9, section 3.1.4]. Obviously, the compositions $\mathsf{M} \circ d$ and $d \circ \mathsf{M}$ are given by $\mathsf{M}(\nabla \bullet \ )$ and $\nabla \bullet \mathsf{M}$ respectively. Having these abstract formulas in hand, we can directly use the form of the action $\bullet$ given by equation (14) to show explicitly that the operators vanish. The computation becomes tedious soon however. Therefore, we rather prove the vanishing by analyzing terms that might occur in formulae for these operators.

The first observation is that the principal (third order) part of operators $\mathsf{M}(\nabla \bullet \ )$ and $\nabla \bullet \mathsf{M}$ vanishes in the torsion-free case, and thus $\mathsf{M} \circ d$ and $d \circ \mathsf{M}$ are operators of order at most one with the curvature in their leading part. This follows from the classification of invariant operators in the flat case – it is well known that there are no operators between the respective bundles, see e.g. [10]. It can be also checked directly by use of (14) for the first two terms in (15). This is doable since we may freely commute the derivatives. It can be also partially deduced from the computation of the principal part of $\Box_{ABCD}$ below.

Since there are no terms of order three in $\mathsf{M} \circ d(\varphi)$ and $d \circ \mathsf{M}(\psi)$ for each φ and ψ, the operators are given by projections to target bundles of a linear combination of terms of the form $R \cdot \nabla \phi$ and $\nabla R \cdot \phi$, where $\phi = \varphi$ and $\phi = \psi$ respectively. According to the description of curvature in section 1.1, the operators may be also written as projections of a linear combination of terms of the form $W \cdot \nabla \phi, \nabla W \cdot \phi$ and $\mathsf{P} \cdot \nabla \phi, \nabla \mathsf{P} \cdot \phi$, where the tensor P_{ab} is symmetric, and the symmetries of tensor $W_{ab}{}^d{}_c$ and exterior derivative of the Rho-tensor are described in (2) and (3) respectively. Since $\mathsf{M} \circ d$ and $d \circ \mathsf{M}$ are natural by construction, given a point x we can compute $M(d\varphi)(x)$ and $d(\mathsf{M}\psi)(x)$ for each φ, ψ in terms of any distinguished connection. So let us choose a distinguished connection ∇_a such that $\mathsf{P}_{(ab)} = 0$ and also $\nabla_{(a}\mathsf{P}_{bc)} = 0$ at x. Its existence follows from the transformation formulas of the Rho-tensor and its covariant derivative. One can actually demand that the total symmetrizations of all derivatives of P_{ab} vanish at x. This is a general feature which refers to the so called normal scale for a parabolic geometry, see [1, theorem 5.1.12]. Evidently, $M(d\varphi)(x)$ and $d(\mathsf{M}\psi)(x)$ for such a connection are then given by projections of terms which contain either tensor $W_{ab}{}^d{}_c$ or tensor $(d^\nabla \mathsf{P})_{abc} = \nabla_{[a}\mathsf{P}_{b]c}$. However, by (2) and (3) both tensors are symmetric in three unprimed spinor indices while the Young diagrams of target bundles has only two columns - recall that the target bundles of the projections are the two bundles in the bottom of composition series (9). Thus all terms vanish under the projections. Since x, φ, ψ were arbitrary, we conclude that the invariant operators $\mathsf{M} \circ d$ and $d \circ \mathsf{M}$ are the zero operators.

Now we know that $\square_{ABCD}$ is invariant. To finish the proof, we need to show that its principal part is given by a nonzero scalar multiple of $\widetilde{\square}_{ABCD}$. First, we find even a full formula in an abstract form. By our construction, $\square_{ABCD}$ is given by a formula in the bottom slot of the section of tractor bundle (9) which is given by the action of $\mathcal{C}^2 \circ (\mathcal{C} - 4) \circ (\mathcal{C} + 4)$ on the section with σ in the top slot and zeros elsewhere. Acting with $\mathcal{C}^2$ first, by formula (6) for $\mathcal{C}$ we get zeros everywhere except for the middle slot and the slot below, where up to the factor 4 we get $(\nabla \bullet \nabla \bullet \sigma)_{1,2} \mp 2(\mathsf{P} \bullet \bullet \sigma)_{1,2}$ and $\mathsf{P} \bullet \bullet \nabla \bullet \sigma$ respectively. Further, acting with $(\mathcal{C} - 4) \circ (\mathcal{C} + 4)$ we get zeros everywhere except for the bottom slot (in the slot above the bottom there is $\mathsf{M}(\,d\sigma) = 0$), and it is easy to deduce that, up to a scalar multiple, the formula there reads as

$$\square\sigma = \nabla \bullet \nabla \bullet (\nabla \bullet \nabla \bullet \sigma)_1 + \nabla \bullet \nabla \bullet (\nabla \bullet \nabla \bullet \sigma)_2 - 2\mathsf{P} \bullet \bullet (\nabla \bullet \nabla \bullet \sigma)_1$$

$$+2\mathsf{P} \bullet \bullet (\nabla \bullet \nabla \bullet \sigma)_2 - 2\nabla \bullet \nabla \bullet (\mathsf{P} \bullet \bullet \sigma)_1 + 2\nabla \bullet \nabla \bullet (\mathsf{P} \bullet \bullet \sigma)_2.$$

$$(16)$$

A detailed derivation of this formula can be found in [9, section 3.1.6]. Thus we see that the principal part is given by the sum of the two possible paths from the top slot to the bottom slot. Now we make the corresponding two terms explicit by a multiple use of formula (14) for $\bullet$. We directly get that the term corresponding to the path through slot $(\ )_1$ is given by $8\nabla_{J'A_1}\nabla_{I'B_1}\nabla^{(I'}_{(B_2}\nabla^{J')}_{A_2)}\sigma_{\ddot{\mathbf{A}}\ddot{\mathbf{B}}}$, followed by the (unique) projection to the target bundle. Recall that the target bundle is the bundle in the bottom slot of (9), i.e.

$$\boxed{\begin{smallmatrix} \\ \end{smallmatrix}}\mathcal{E}_{\mathbf{AB}}[-k] \subset \mathcal{E}_{(\mathbf{AB})}[-k] \qquad (17)$$

and thus the projection is given by alternation in $A_1, \ldots, A_k$ and in $B_1, \ldots, B_k$, followed by symmetrization in $\mathbf{A}$ and $\mathbf{B}$ and projection to the joint kernel of alternations in more than k inputs. Note that we may freely commute the derivatives since we are only interested in the principal part. In particular, the symmetry of primed indices of covariant derivatives translates to unprimed indices and vice versa. Hence expanding the symmetrizations in the previous formula, we get an equivalent formula

$$4\nabla_{J'A_1}\nabla^{J'}_{A_2}\nabla_{I'B_1}\nabla^{I'}_{B_2}\sigma_{\ddot{\mathbf{A}}\ddot{\mathbf{B}}} + 4\nabla_{J'A_1}\nabla^{I'}_{A_2}\nabla_{I'B_1}\nabla^{J'}_{B_2}\sigma_{\ddot{\mathbf{A}}\ddot{\mathbf{B}}}.$$

Moreover, since the explicit form of the isomorphism $\mathcal{E}^{[A'B']} \cong \mathcal{E}[-1]$ reads as $v^{[A'B']} = -1/2 \cdot v_{I'}{}^{I'}\epsilon^{A'B'}$, the second summand is the previous display is equivalent to $-\nabla_{J'A_1}\nabla^{J'}_{A_2}\nabla_{I'B_1}\nabla^{I'}_{B_2}\sigma_{\ddot{\mathbf{A}}\ddot{\mathbf{B}}}$. Therefore, the principal part of $\nabla \bullet \nabla \bullet (\nabla \bullet \nabla \bullet \sigma)_1$ is given by the projection of $3\nabla_{J'A_1}\nabla^{J'}_{A_2}\nabla_{I'B_1}\nabla^{I'}_{B_2}\sigma_{\ddot{\mathbf{A}}\ddot{\mathbf{B}}}$, and thus it coincides with $3 \cdot \widetilde{\square}_{ABCD}$. Similarly, we deduce directly by a multiple use of (14) that the leading term corresponding to the path through slot $(\ )_2$ is given by

$$4\nabla_{I'A_1}\nabla^{I'}_{A_2}\nabla_{J'B_1}\nabla^{J'}_{B_2}\sigma_{\ddot{\mathbf{A}}\ddot{\mathbf{B}}} - 2k(-1)^k\nabla_{I'A_1}\nabla^{I'}_{B_1}\nabla_{J'[B_2}\nabla^{J'}_{B_3}\sigma_{\ddot{\mathbf{B}}A_2]\ddot{\mathbf{A}}}, \qquad (18)$$

followed by the projection to target bundle (17). The second term can be rewritten by expanding the displayed alternation with respect to A_2 and $\ddot{\mathbf{B}}$ as follows

$$\nabla_{J'[B_2}\nabla^{J'}_{B_3}\sigma_{\ddot{\mathbf{B}}A_2]\ddot{\mathbf{A}}} = \tfrac{1}{k}(-1)^{k-1}\nabla_{J'A_2}\nabla^{J'}_{[B_2}\sigma_{\ddot{\mathbf{B}}]\ddot{\mathbf{A}}} + \tfrac{1}{k}(-1)^{k-2}\nabla_{J'[B_2}\nabla^{J'}_{|A_2|}\sigma_{\ddot{\mathbf{B}}]\ddot{\mathbf{A}}}$$

$$+\tfrac{k-2}{k}\nabla_{J'[B_2}\nabla^{J'}_{B_3}\sigma_{\ddot{\mathbf{B}}]A_2\ddot{\mathbf{A}}}.$$

Since $\sigma_{\mathbf{\ddot{A}\ddot{B}}}$ is a section of the bundle which corresponds to Young diagram with two columns of height $k-2$, the last term on the right hand side in the previous equation vanishes under the alternation over $A_2, \ldots, A_k$. Therefore, we can substitute $\nabla_{J'[B_2}\nabla^{J'}_{B_3}\sigma_{\mathbf{\ddot{B}}A_2]\mathbf{\ddot{A}}}$ in (18) by $\frac{2}{k}(-1)^{k-1}\nabla_{J'[A_2}\nabla^{J'}_{B_2]}\sigma_{\mathbf{\ddot{A}\ddot{B}}}$, and thus we conclude that the principal part of $\nabla \bullet \nabla \bullet (\nabla \bullet \nabla \bullet \sigma)_2$ is given by the projection of

$$4\nabla_{I'A_1}\nabla^{I'}_{A_2}\nabla_{J'B_1}\nabla^{J'}_{B_2}\sigma_{\mathbf{\ddot{A}\ddot{B}}} + 4\nabla_{I'[A_1}\nabla^{I'}_{B_1]}\nabla_{J'[A_2}\nabla^{J'}_{B_2]}\sigma_{\mathbf{\ddot{A}\ddot{B}}}$$

to the target bundle. Now we observe that this is actually the same formula as in the case of the first path up to a commutation of derivatives. Hence by the same reasons as above, the principal part of $\nabla \bullet \nabla \bullet (\nabla \bullet \nabla \bullet \sigma)_2$ coincides with $3 \cdot \widetilde{\square}_{ABCD}$. The principal part of $\square_{ABCD}$ then equals $6 \cdot \widetilde{\square}_{ABCD}$.

The proposition shows in particular that in the torsion-free case there exist a curved analogue for each of the nonstandard operators. Hence it gives an alternative proof of Theorem 5.1 of [5]. Moreover, our construction directly yields a formula for each of these curved analogues, cf. formula (16). Making this formula explicit and using the symmetries of the Rho-tensor and its exterior derivative, we get the following.

Corollary 2.5 *The Grassmannian nonstandard operator can be written as the projection of an operator $d \circ A \circ d$ to bundle (17), where A is a noninvariant operator*

$$A_{AB} : \mathcal{E}^{A'} \otimes \begin{array}{c}\boxed{}\\[-3pt]\vdots\\[-3pt]\boxed{}\end{array} \mathcal{E}_{\mathbf{\dot{A}\ddot{B}}}[-k+2] \to \mathcal{E}^{A'} \otimes \begin{array}{c}\boxed{}\\[-3pt]\vdots\\[-3pt]\boxed{}\end{array} \mathcal{E}_{\mathbf{A\dot{B}}}[-k+1]$$

which is given by

$$
\begin{aligned}
(A\mu)^{A'}_{\mathbf{A\dot{B}}} &= 2\nabla_{J'A_1}\nabla^{(A'}_{A_2}\mu^{J')}_{\mathbf{\dot{B}\ddot{A}}} + 2\nabla_{J'A_1}\nabla^{(A'}_{B_2}\mu^{J')}_{\mathbf{A\ddot{B}}} + 2\nabla^{A'}_{B_2}\nabla_{J'A_1}\mu^{J'}_{\mathbf{A\dot{B}}} \\
&\quad + \nabla^{A'}_{A_1}\nabla_{J'B_2}\mu^{J'}_{\mathbf{A\ddot{B}}} - \nabla^{A'}_{A_1}\nabla_{J'A_2}\mu^{J'}_{\mathbf{\dot{B}\ddot{A}}} + 16\mathsf{P}^{(A'J')}_{(B_2A_1)}\mu_{J'\mathbf{A\ddot{B}}} + 8\mathsf{P}^{J'}_{J'A_1A_2}\mu^{A'}_{\mathbf{\dot{B}\ddot{A}}} \\
&\quad + 8\mathsf{P}^{J'}_{J'[A_1B_2]}\mu^{A'}_{\mathbf{A\ddot{B}}}.
\end{aligned}
$$

Applying the explicit form of the action $\bullet$ from (14) to formula (16), we directly get that $(\square f)_{\mathbf{AB}}$ is given by the action of

$$
\begin{aligned}
\nabla_{A_1A_2B_1B_2} &= 8\nabla_{I'(A_1}\nabla_{|J'|B_1)}\nabla^{(I'}_{(A_2}\nabla^{J')}_{B_2)} + 2\nabla_{I'A_1}\nabla^{I'}_{A_2}\nabla_{J'B_1}\nabla^{J'}_{B_2} \\
&\quad + 2\nabla_{I'B_1}\nabla^{I'}_{B_2}\nabla_{J'A_1}\nabla^{J'}_{A_2} + 4\nabla_{I'[A_1}\nabla^{I'}_{B_1]}\nabla_{J'[A_2}\nabla^{J'}_{B_2]} \\
&\quad - 16\mathsf{P}_{I'(A_1|J'|B_1)}\nabla^{(I'}_{(A_2}\nabla^{J')}_{B_2)} + 4\mathsf{P}_{I'}{}^{I'}{}_{A_1A_2}\nabla_{J'B_1}\nabla^{J'}_{B_2} \\
&\quad + 4\mathsf{P}_{I'}{}^{I'}{}_{B_1B_2}\nabla_{J'A_1}\nabla^{J'}_{A_2} + 8\mathsf{P}_{I'}{}^{I'}{}_{[A_1B_1]}\nabla_{J'[A_2}\nabla^{J'}_{B_2]} \\
&\quad - 16\nabla_{I'(A_1}\nabla_{|J'|B_1)}\mathsf{P}^{(I'J')}_{(A_2B_2)} + 4\nabla_{I'A_1}\nabla^{I'}_{A_2}\mathsf{P}_{J'}{}^{J'}{}_{B_1B_2} \\
&\quad + 4\nabla_{I'B_1}\nabla^{I'}_{B_2}\mathsf{P}_{J'}{}^{J'}{}_{A_1A_2} + 8\nabla_{I'[A_1}\nabla^{I'}_{B_1]}\mathsf{P}_{J'}{}^{J'}{}_{[A_2B_2]},
\end{aligned}
$$

on a section $f_{\mathbf{\ddot{A}\ddot{B}}}$. Now we simplify this formula by using symmetries (3) of the covariant derivative of the Rho-tensor. Namely, we can replace each tensor $\nabla\mathsf{P}$ by its totally symmetric part since all other components are symmetric in three unprimed spinor indices and thus all terms containing them vanish when projected to target (17). Hence the we get the identity

$$-2\nabla_{J'B_1}\mathsf{P}^{(I'J')}_{(A_2B_2)} + \nabla^{I'}_{A_2}\mathsf{P}_{J'}{}^{J'}{}_{B_1B_2} + \nabla^{I'}_{B_1}\mathsf{P}_{J'}{}^{J'}{}_{[A_2B_2]} = 0$$

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Applying this equation to the previous formula we conclude that the sum of terms of type $\nabla\nabla\mathsf{P}f$ is equal to the sum of terms of type $\mathsf{P}\nabla\nabla f$ and that the lower order terms may be written as

$$\nabla_{I'A_1}(-16\mathsf{P}^{(I'J')}_{(A_2B_2)}\nabla_{J'B_1} + 8\mathsf{P}^{J'}_{J'B_1B_2}\nabla^{I'}_{A_2} + 8\mathsf{P}^{J'}_{J'[A_2B_2]}\nabla^{I'}_{B_1})f_{\ddot{\mathbf{A}}\ddot{\mathbf{B}}}$$
$$+\nabla_{I'B_1}(-16\mathsf{P}^{(I'J')}_{(B_2A_2)}\nabla_{J'A_1} + 8\mathsf{P}^{J'}_{J'A_1A_2}\nabla^{I'}_{B_2} + 8\mathsf{P}^{J'}_{J'[B_2A_2]}\nabla^{I'}_{A_1})f_{\ddot{\mathbf{A}}\ddot{\mathbf{B}}}.$$

Rewriting the leading part accordingly, the result follows by applying formulas for the exterior derivative $(df)^{A'}_{\ddot{\mathbf{A}}\ddot{\mathbf{B}}} = \nabla^{A'}_{[A_2}f_{\ddot{\mathbf{A}}]\ddot{\mathbf{B}}}$ and $(d\nu)_{\mathbf{A}\mathbf{B}} = \nabla_{I'[A_1}\nu^{I'}_{\ddot{\mathbf{A}}]\mathbf{B}} + \nabla_{I'[B_1}\nu^{I'}_{\ddot{\mathbf{B}}]\mathbf{A}}$.

Remark 2.6 The torsion-freeness of the structure is important for the proof of proposition 2.4. In the case of nonvanishing torsion, the Weyl curvature does not lie in the irreducible bundle (2) but consists of the harmonic part and some other irreducible components which can be expressed in terms of torsion. Of course, also the derivative of the Rho-tensor does not lie in (3). The consequence of these facts is that the operators $\mathsf{M}\circ d$ and $d\circ\mathsf{M}$ do not vanish. Namely, it is easy to show that they both are second order operators with the torsion in their leading part. Our construction breaks down therefore. However, this does not mean that curved analogues of the nonstandard operators do not exist in such a case. Indeed, it is proved in [9, Section 3.2] that the operator which acts on functions exists also in the presence of a nonzero torsion. On the other hand, it is proved there that this operator cannot be written as in corollary 2.5, i.e. as a composition of operators with the exterior derivatives in the beginning and at the end.

3 Weak invariance of nonstandard operators

The invariance of operators $\square_{ABCD}$ constructed above obviously depends on the vanishing of operators $\mathsf{M}\circ d$ and $d\circ\mathsf{M}$ in the torsion-free case. The first step of the proof of the vanishing of these operators was an observation that their leading (third order) terms can be rewritten in terms of the first derivative and the curvature. Hence we need to commute covariant derivatives in order to prove the invariance of operators $\square_{ABCD}$. This shows in turn that if we replace the distinguished connection ∇ by the coupled distinguished tractor connection, defined by the Leibnitz rule in the usual way, then the formula obtained from the construction does not define invariant operator on tractor bundles. The transformation of such operator will consist of terms containing tractor curvature in general. A natural question now is whether there exist a formula for $\square_{ABCD}$ which is universal in the sense that it defines invariant operators also on tractor bundles or there is no such formula, i.e. whether $\square_{ABCD}$ is strongly invariant or not. We will mainly use a slightly different notion of strong invariance. Namely, an operator is strongly invariant in the sense of [12] if it factors through semi-holonomic jets. In the dual picture, it means that it is induced by a homomorphism of semi-holonomic Verma modules. Let us remark that such an operator then translates to tractor bundles, see [14]. We will prove in this section that operators $\square_{ABCD}$ are not strongly invariant in the algebraic sense. In the subsequent remark, we will argue that the operators are also not strongly invariant in the sense of the existence of a universal formula.

3.1 Nonstandard operators are not strongly invariant

Let us digress to the case of locally flat structures for a moment. It is well known that then the jet bundles are associated to the Cartan bundle. Hence invariant differential operators are in a bijective correspondence with homomorphisms between jet prolongations of representations which induce the bundles in question. According to the definition of the flat nonstandard operators $\widetilde{\square}_{ABCD}$, the respective homomorphisms are $\mathfrak{p}$-homomorphisms $\Phi : \mathcal{J}^4(\mathbb{V}_{k-2}) \to \mathbb{V}_k$, where for each $2 \le k \le n$ we set

$$\mathbb{V}_k := \overset{\updownarrow}{}{}_k \; \boxed{} \; \mathbb{R}^{n*}[-k]$$

for the module inducing bundle (17), and where $\mathcal{J}^4(\mathbb{V}_k)$ denotes a module inducing the fourth jet prolongation of this bundle. By the description of the flat nonstandard operators in section 1.5, for each convenient k the map Φ is given by a composition of (up to a multiple) unique $\mathfrak{g}_0$-homomorphism $\phi : S^4\mathfrak{g}_{-1}^* \otimes \mathbb{V}_{k-2} \to \mathbb{V}_2 \otimes \mathbb{V}_{k-2}$, which in terms of abstract indices reads

$$\phi(\omega)_{ABCD\mathbf{EF}} = \frac{1}{2}(\omega_{I'[AB]}{}^{I'}{}_{J'[CD]}{}^{J'}{}_{\mathbf{EF}} + \omega_{I'[CD]}{}^{I'}{}_{J'[AB]}{}^{J'}{}_{\mathbf{EF}}) - \omega_{I'[AB|}{}^{I'}{}_{J'|CD]}{}^{J'}{}_{\mathbf{EF}}, \quad (19)$$

with a unique projection $\mathbb{V}_2 \otimes \mathbb{V}_{k-2} \to \mathbb{V}_k$. We will manifest in the course of the forthcoming proof that Φ is indeed a $\mathfrak{p}$-homomorphism.

In contrary to ordinary jet bundles, the semi-holonomic jet bundles are associated to the Cartan bundle also on structures with nonzero curvature. The respective representation, denoted by $\bar{\mathcal{J}}^k()$, is called the semi-holonomic jet prolongation. The dual module is the so called semi-holonomic Verma module. For more details see [13] and [12]. The operators which are strongly invariant in the algebraic sense are exactly the operators which are induced by homomorphisms between these modules. We will prove that operator $\square_{ABCD}$ does not belong to them. That is, there does not exist any $\mathfrak{p}$-homomorphism $\widetilde{\Phi} : \bar{\mathcal{J}}^4(\mathbb{V}_{k-2}) \to \mathbb{V}_k$ such that its restriction to the holonomic jets $\mathcal{J}^4(\mathbb{V}_{k-2}) \subset \bar{\mathcal{J}}^4(\mathbb{V}_{k-2})$ coincides with Φ (this condition says in other words that $\square_{ABCD}$ coincides with $\widetilde{\square}_{ABCD}$ on locally flat structures).

Proposition 3.1 *The Grassmannian nonstandard operators are not strongly invariant.*

We shall prove that there exists no $\mathfrak{p}$-homomorphism $\widetilde{\Phi}$ which extends Φ. We prove it in two steps. By [13, Lemma 5.8], a $\mathfrak{g}_0$-homomorphism $\widetilde{\Phi}$ is $\mathfrak{p}$-homomorphism if and only if it factors through $\mathcal{J}^4(\mathbb{V}_{k-2}) \to \otimes^4\mathfrak{g}_{-1}^* \otimes \mathbb{V}_{k-2}$ and it vanishes on the image of $\otimes^3\mathfrak{g}_{-1}^* \otimes \mathbb{V}_{k-2}$ under the action of $\mathfrak{g}_1$. Hence we analyze first the space of $\mathfrak{g}_0$-homomorphisms $\widetilde{\Phi} : \otimes^4\mathfrak{g}_{-1}^* \otimes \mathbb{V}_{k-2} \to \mathbb{V}_k$, and then we describe explicitly the image of the action of $\mathfrak{g}_1$ on $\bar{\mathcal{J}}^4(\mathbb{V}_{k-2})$ and we show that it never lies in $\mathrm{Ker}(\widetilde{\Phi})$ for any $\widetilde{\Phi}$ which lifts Φ.

Step 1. An important observation is that any $\mathfrak{g}_0$-homomorphism $\otimes^4\mathfrak{g}_{-1}^* \otimes \mathbb{V}_{k-2} \to \mathbb{V}_k$ factors through $\mathbb{V}_2 \otimes \mathbb{V}_{k-2}$ and a unique projection $\mathbb{V}_2 \otimes \mathbb{V}_{k-2} \to \mathbb{V}_k$. Therefore, we only need to analyze $\mathfrak{g}_0$-homomorphisms $\otimes^4\mathfrak{g}_{-1}^* \to \mathbb{V}_2$. This map is a complete contraction on the $\mathbb{R}^2$-part of $\mathfrak{g}_{-1}^*$ and thus a linear combination of $c_1(\omega) = \omega_{I'}{}^{I'}{}_{J'}{}^{J'}$, $c_2(\omega) = \omega_{I'J'}{}^{J'I'}$ and $c_3(\omega) = \omega_{I'}{}^{J'I'}{}_{J'}$, which are related by $c_1 + c_2 + c_3 = 0$. Similarly, on the $\mathbb{R}^{n*}$-part of $\mathfrak{g}_{-1}^*$ we denote the three

projections $\otimes^4 \mathbb{R}^{n*} \to \boxplus \mathbb{R}^{n*}$ as follows

$$p_1(\omega)_{\mathbf{A}^2\mathbf{B}^2} = \tfrac{1}{2}(\omega_{A_1 A_2 B_1 B_2} + \omega_{B_1 B_2 A_1 A_2}) - \omega_{[A_1 A_2 B_1 B_2]},$$

$$p_2(\omega)_{\mathbf{A}^2\mathbf{B}^2} = \tfrac{1}{2}(\omega_{A_1 B_1 B_2 A_2} + \omega_{B_1 A_1 A_2 B_2}) - \omega_{[A_1 A_2 B_1 B_2]},$$

$$p_3(\omega)_{\mathbf{A}^2\mathbf{B}^2} = \tfrac{1}{2}(\omega_{A_1 B_2 A_2 B_1} + \omega_{B_1 A_2 B_2 A_1}) - \omega_{[A_1 A_2 B_1 B_2]}.$$

They are obviously related by a similar equation $p_1 + p_2 + p_3 = 0$. Hence the $\mathfrak{g}_0$-homomorphisms $\otimes^4\mathfrak{g}_{-1}^* \to \mathbb{V}_2$ form a vector space of dimension four. We choose $c_i \circ p_j$, $i, j = 1, 2$ as a basis and we set $\Phi_{ii} := (c_i \circ p_i) \otimes \mathrm{id}$ and $\Phi_{ij} := -2(c_i \circ p_j) \otimes \mathrm{id}$ for $i \neq j$. Then any $\mathfrak{g}_0$-homomorphism $\widetilde{\Phi} : \otimes^4\mathfrak{g}_{-1}^* \otimes \mathbb{V}_{k-2} \to \mathbb{V}_k$ is given by a linear combination $K\Phi_{11} + L\Phi_{12} + M\Phi_{21} + N\Phi_{22}$, followed by the unique projection to $\mathbb{V}_k$. Moreover, the uniqueness in symmetric case implies that the restriction of each Φ_{ij} to $S^4\mathfrak{g}_{-1}^* \otimes \mathbb{V}_{k-2}$ is a multiple of ϕ, which is given by (19) and which defines the holonomic map Φ. Precisely, it it is easy to compute that upon the restriction to $\mathcal{J}^4(\mathbb{V}_{k-2})$ all maps Φ_{ij} coincide with ϕ. This shows that $\widetilde{\Phi}$ covers Φ if and only if it is given by a linear combination of maps Φ_{ij} such that the coefficients satisfy $K + L + M + N = 1$.

Step 2. By [13, Lemma 5.10], the action of $Z \in \mathfrak{g}_1$ on $\psi \in \otimes^3\mathfrak{g}_{-1}^* \otimes \mathbb{V}_{k-2}$, regarded as a $\mathfrak{g}_0$-submodule of $\bar{\mathcal{J}}^4(\mathbb{V}_{k-2})$, is given by

$$(Z \cdot \psi)(X_1, \cdots, X_4) = \sum_{1 \leq i \leq 4} [Z, X_i] \bullet \psi(X_1, \cdots, \hat{X}_i, \cdots, X_4)$$
$$- \sum_{1 \leq i < j \leq 4} \psi(X_1, \cdots, [[Z, X_i], X_j]\hat{X}_j, \cdots, X_4) \tag{20}$$

In terms of abstract indices with conventions from section 1.1, the element $[Z, X_i]$ of $\mathfrak{g}_0$ is given by $[Z, X_i] = (Z_I^{A'}(X_i)_{B'}^I, Z_A^{I'}(X_i)_{I'}^B)$, and its action on $v^{A'} \in \mathbb{R}^2$, $v_A \in \mathbb{R}^{q*}$, $\sigma \in \mathbb{R}[w]$ and its adjoint action on $X_j \in \mathfrak{g}_{-1}$ reads as follows

$$[Z, X_i] \bullet v^{A'} = Z_I^{A'}(X_i)_{I'}^I v^{I'},$$

$$[Z, X_i] \bullet v_A = Z_A^{I'}(X_i)_{I'}^I v_I,$$

$$[Z, X_i] \bullet \sigma = w Z_I^{I'}(X_i)_{I'}^I \sigma,$$

and

$$[[Z, X_i], X_j]_{A'}^A = -Z_I^{I'}(X_i)_{I'}^A(X_j)_{A'}^I - Z_I^{I'}(X_i)_{A'}^I(X_j)_{I'}^A.$$

Now we apply these equations to (20). It is easy to see that for $Z_{a_i} = Z_{A_i}^{A'} \in \mathfrak{g}_{-1}^*$ and $\psi_{a_1 a_2 a_3 \mathbf{E}\mathbf{F}} \in \otimes^3\mathfrak{g}_{-1}^* \otimes \mathbb{V}_{k-2}$ the terms $[Z, X_i] \bullet \psi$ appearing in the first sum are given by

$$([Z, X_i] \bullet \psi)_{a_1 a_2 a_3 a_4 \mathbf{E}\mathbf{F}} = (k - 2) Z_{A_i}^{A'} \psi_{a_1 \cdots \hat{a}_i \cdots a_4 \mathbf{E}\mathbf{F}}$$
$$+ (k - 2) Z_{E_1}^{A'} \psi_{a_1 \cdots \hat{a}_i \cdots a_4 A_i \dot{\mathbf{E}}\mathbf{F}} + (k - 2) Z_{F_1}^{A'} \psi_{a_1 \cdots \hat{a}_i \cdots a_4 \mathbf{E}A_i \dot{\mathbf{F}}},$$

Now it is an easy observation that the unprimed spinor indices displayed on the right-hand side of the previous formula lie in

$$\overset{\uparrow}{\underset{\downarrow}{k-2}} \boxed{\begin{array}{cc} & \\ \vdots & \\ & \end{array}} \mathbb{R}^{n*}.$$

It means that each term $[Z, X_i] \bullet \psi$ vanishes under the projection to $\mathbb{V}_k$, and thus only the second sum in (20) remains modulo $\mathrm{Ker}(\widetilde{\Phi})$. For $\psi = \omega \otimes v$, where $\omega \in \otimes^3 \mathfrak{g}_{-1}^*$ and $v \in \mathbb{V}_{k-2}$, it reads as follows

$$(Z \cdot (\omega \otimes v))_{ABCD}^{A'B'C'D'} = (Z_B^{A'} \omega_{ACD}^{B'C'D'} + Z_A^{B'} \omega_{BCD}^{A'C'D'} + Z_C^{A'} \omega_{BAD}^{B'C'D'} + Z_A^{C'} \omega_{BCD}^{B'A'D'}$$
$$+ Z_D^{A'} \omega_{BCA}^{B'C'D'} + Z_A^{D'} \omega_{BCD}^{B'C'A'} + Z_C^{B'} \omega_{ABD}^{A'C'D'} + Z_B^{C'} \omega_{ACD}^{A'B'D'} + Z_D^{B'} \omega_{ACB}^{A'C'D'}$$
$$+ Z_B^{D'} \omega_{ACD}^{A'C'B'} + Z_D^{C'} \omega_{ABC}^{A'B'D'} + Z_C^{D'} \omega_{ABD}^{A'B'C'}) \otimes v.$$

In order to show that for some ω it does not lie in $\mathrm{Ker}(\widetilde{\Phi})$ for any lift $\widetilde{\Phi}$, we express its image under each Φ_{ij}. A straightforward computation yields

$$c_1(Z \cdot \omega)_{ABCD} = -Z_{I'C}(\omega_{BADJ'}^{I'J'} + \omega_{J'ABD}^{I'J'})$$
$$+ Z_{I'D}(-\omega_{BCAJ'}^{I'J'} + \omega_{ACBJ'}^{I'J'} + \omega_{J'ABC}^{J'I'})$$

and

$$c_2(Z \cdot \omega)_{ABCD} = Z_{I'C}(\omega_{J'BAD}^{J'I'} + \omega_{J'ABD}^{J'I'})$$
$$+ Z_{I'D}(\omega_{J'BCA}^{J'I'} + \omega_{J'ACB}^{I'J'} - \omega_{J'ABC}^{I'J'}).$$

Now it is easy to see that both contractions vanish provided that ω is symmetric. And since holonomic homomorphism Φ factors through a complete contraction, it also vanishes on the image of the action of $\mathfrak{g}_1$ on $S^3\mathfrak{g}_{-1}^* \otimes \mathbb{V}_{k-2}$. This shows that Φ is indeed a $\mathfrak{p}$-homomorphism. On the other hand, the contractions c_1, c_2 for a nonsymmetric ω are nonzero in general. Namely, if we set

$$\omega_{ABC}^{A'B'C'} = \epsilon^{A'B'} \bar{\omega}_{ABC}^{C'} \in \mathrm{Ker}(\mathcal{E}_{[AB]C}^{C'}[-1] \to \mathcal{E}_{[ABC]}^{C'}[-1]) \subset \otimes^3 \mathfrak{g}_{-1}^*,$$

then they are equal to

$$c_1(Z \cdot \bar{\omega})_{ABCD} = -3Z_{I'D}\bar{\omega}_{ABC}^{I'}$$

$$c_2(Z \cdot \bar{\omega})_{ABCD} = -3Z_{I'D}\bar{\omega}_{BCA}^{I'}.$$

Now it is easy to compute that the compositions with projections p_1, p_2 yield

$$p_1 \circ c_1(Z \cdot \bar{\omega}) = -2p_2 \circ c_1(Z \cdot \bar{\omega}) = -2p_1 \circ c_2(Z \cdot \bar{\omega}) = p_2 \circ c_2(Z \cdot \bar{\omega}).$$

This fact shows that for $\bar{\psi} := \bar{\omega} \otimes v$ the element $Z \cdot \bar{\psi}$ has the same image under all maps Φ_{ij}. Hence the image of $Z \cdot \bar{\psi}$ under a lift $\widetilde{\Phi}$ of Φ, which is given by a linear combination of maps Φ_{ij} with coefficients K, L, M, N, is given by the projection of
$$(K + L + M + N)\Phi_{11}(Z \cdot \bar{\psi}) = \Phi_{11}(Z \cdot \bar{\psi})$$

to $\mathbb{V}_k$. In particular, it does not depend on the coefficients K, L, M, N and it is not zero. Precisely, we get

$$\widetilde{\Phi}(Z \cdot \bar{\psi})_{\mathbf{AB}} = \frac{1}{2}(Z_{I'B_2}\bar{\psi}_{A_1A_2B_1\mathbf{\ddot{A}\ddot{B}}}^{I'} + Z_{I'A_2}\bar{\psi}_{B_1B_2A_1\mathbf{\ddot{A}\ddot{B}}}^{I'}) \neq 0.$$

Remark 3.2 A careful computation reveals that the maps Φ_{ij} coincide on whole image of $\otimes^3 \mathfrak{g}_{-1}^* \otimes \mathbb{V}_{k-2}$ in $\otimes^4 \mathfrak{g}_{-1}^* \otimes \mathbb{V}_{k-2}$ under the action of $\mathfrak{g}_1$. Since the action of $\mathfrak{g}_1$ gives exactly the transformation of the four-fold covariant derivative which is algebraic and linear in the one-form Υ describing the change $\nabla \to \hat{\nabla}$, the coincidence of maps Φ_{ij} shows in turn that the projection of the algebraic linearized transformation of $\nabla^4 f$ to $\mathbb{V}_k$ does not depend on the succession of covariant derivatives. Indeed, acting with $\square_{ABCD}$ on a tractor bundle, the leading terms with different orders of derivatives differ by terms of the form $\nabla^2 \Omega$, $\nabla \Omega \nabla$ and $\Omega \nabla^2$ where Ω is the tractor curvature. And it is easy to prove that projections to the target bundle of all these terms are invariant. Hence any formula for $\square_{ABCD}$ will have the same linearized transformation of the leading part as the formula obtained from the curved Casimir construction. If we trace where we commuted derivatives back in the proof of proposition 2.4, we conclude that this transformation (described by an one form Υ) is given by the projection of

$$\Upsilon \nabla \Omega + \Upsilon \Omega \nabla.$$

Moreover, it is easy to see that this cannot be cancelled by transformations of lower order terms and thus it is contained in the transformation of any formula for $\square_{ABCD}$. A straightforward computation shows that the action of this curvature expression on the standard tractor bundle vanishes if and only if the harmonic part of the Cartan curvature vanishes, i.e. in the flat case. Hence there exists no universal formula for $\square_{ABCD}$ which would define invariant operator between tractor bundles on manifolds with a general Grassmannian structure.

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