COMPLEMENTS ON ENRICHED PRECATEGORIES

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ABSTRACT. We make some remarks on the foundations of the homotopy theory of enriched precategories, as exposed in Carlos Simpson's book "Homotopy theory of higher categories".

The main goal of this note is to present an alternative point of view on some results from Carlos Simpson's book "Homotopy theory of higher categories" [11]. For a part of these results, the alternative approach is already hinted at by Simpson. Save section 3 below, we target essentially the content of chapters 9-11, sections 12.1-12.3, 12.6 and some facts scattered through chapters 13 and 14. There is one major exception: we do not treat here the so-called injective model structure on the category of enriched precategories with fixed set of objects.

In section 1 we introduce one of the two categories of interest to us and we highlight some of its properties. In section 2 we first address the Reedy model structure on enriched precategories with fixed set of objects and study the behaviour of this model structure under change of diagram and base category. Next we address the projective model structure on enriched precategories with fixed set of objects and compare it with the Reedy model structure. Section 3 reviews Lurie's proof [7] of the Quillen equivalence between the projective model structure on enriched precategories and that of enriched categories, in the (critical) fixed set of objects case. We observe that his result holds under weaker assumptions on the base category. In section 4 we recall the construction of the category of enriched precategories–the other category of interest to us, and we remark that it is a bifibration over the category of sets. Section 5 introduces the fibred Reedy model structure on the category of enriched precategories and singles out a certain weak factorization system on it. In section 6 we introduce the fibred projective model structure on the category of enriched precategories. In the appendix we recall a couple of results concerning left Bousfield localization.

Notation. The terminal object of a category, when it exists, is denoted by $*$.

Independent from the results of this note, we make below a list of some facts from [11] we wish to understand better in the future.

(1) On page 291, why is the sentence "Furthermore,...(see Lemma 10.3.2)" true?

(2) One may compare (the proof of) 13.7.3 with the paragraph on page 447 "A map...Thus, p is a global weak equivalence."

(3) 14.3.5 is used in the proof of 19.2.1(PGM6). It is not mentioned in the statement of 19.2.1 that the class of weak equivalences of $\mathcal{M}$ is closed under transfinite compositions.

(4) In the proof of 18.7.1, on page 438, the maps $f|_{\mathcal{B} \times \{v_1\}}$ and q do not seem to have the same target.

(5) In the proof of 19.3.1, why is the left vertical map in the diagram a cofibration?

1. THE CATEGORY $\mathbf{M}_*^{\mathcal{C}^{op}}$

Let $\mathcal{C}$ be a small Reedy category and let $F^n\mathcal{C}$ be the n-filtration of $\mathcal{C}$ [6, 15.1.22]; then $F^0\mathcal{C}$ is a discrete category [6, 15.1.23]. We denote the inclusion $F^0\mathcal{C} \subset \mathcal{C}$ by σ_0. Let $\mathbf{M}$ be a category. The restriction functor $\sigma_0^* : \mathbf{M}^{\mathcal{C}^{op}} \to \mathbf{M}^{(F^0\mathcal{C})^{op}}$ has a left adjoint $\sigma_{0!}$ provided that $\mathbf{M}$ has coproducts.

Definition 1.1. *Let $\mathbf{M}$ be a category with terminal object. We denote by $\mathbf{M}_*^{\mathcal{C}^{op}}$ the full subcategory of $\mathbf{M}^{\mathcal{C}^{op}}$ on objects $\mathbf{X}$ such that $\sigma_0^*\mathbf{X} = *$. We let K be the inclusion functor $\mathbf{M}_*^{\mathcal{C}^{op}} \subset \mathbf{M}^{\mathcal{C}^{op}}$.*

Some facts about $\mathbf{M}_*^{\mathcal{C}^{op}}$ 1.2. (a) K creates colimits indexed by connected digrams and it has a left adjoint r calculated as follows. If $\mathbf{X} \in \mathbf{M}^{\mathcal{C}^{op}}$, then one has a pushout square

$$
\begin{array}{ccc}
\sigma_{0!}\sigma_0^*\mathbf{X} & \longrightarrow & \mathbf{X} \\
\downarrow & & \downarrow \\
\sigma_{0!}\sigma_0^** & \longrightarrow & r\mathbf{X}
\end{array}
$$

(b) $\mathbf{M}_*^{\mathcal{C}^{op}}$ is an accessible category if $\mathbf{M}$ is.

(c) Suppose that $\mathbf{M}$ is a closed category with monoidal product $\otimes$. Write Y^X for the internal hom of two objects X, Y of $\mathbf{M}$. Then $\mathbf{M}^{\mathcal{C}^{op}}$ is tensored, cotensored and enriched over $\mathbf{M}$, with tensor, cotensor and $\mathbf{M}$-hom defined as

$$(A\bigstar\mathbf{X})(c) = A \otimes \mathbf{X}(c)$$

$$(\mathbf{X}^A)(c) = \mathbf{X}(c)^A$$

$$\underline{Map}(\mathbf{X},\mathbf{Y}) = \int_c \mathbf{Y}(c)^{\mathbf{X}(c)}$$

It follows that $\mathbf{M}_*^{\mathcal{C}^{op}}$ is tensored, cotensored and enriched over $\mathbf{M}$, with tensor, cotensor and $\mathbf{M}$-hom defined by the formulae $A\bigstar\mathbf{X} = r(A\bigstar K\mathbf{X})$, $\mathbf{X}^A = (K\mathbf{X})^A$ and $\underline{Map}(\mathbf{X},\mathbf{Y}) = \underline{Map}(K\mathbf{X},K\mathbf{Y})$. The adjunction (r,K) becomes an $\mathbf{M}$-adjunction.

(d) For every small category I there is an isomorphism of categories

$$(\mathbf{M}^I)_*^{\mathcal{C}^{op}} \cong (\mathbf{M}_*^{\mathcal{C}^{op}})^I$$

and these categories are isomorphic in turn to the full subcategory of $\mathbf{M}^{\mathcal{C}^{op}\times I}$ on objects $\mathbf{X}$ having the property that $\mathbf{X}(c_0,i) = *$, for $c_0 \in F^0\mathcal{C}$ and $i \in I$.

(e) (Change of diagram) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between Reedy categories which preserves the 0-filtration. The induced functor $F^* : \mathbf{M}_*^{\mathcal{C}^{op}} \to \mathbf{M}_*^{\mathcal{D}^{op}}$ has a left adjoint $F_!'$ constructed in such a way that it makes the diagram

$$
\begin{array}{ccc}
\mathbf{M}^{\mathcal{C}^{op}} & \underset{F^*}{\overset{F_!}{\rightleftarrows}} & \mathbf{M}^{\mathcal{D}^{op}} \\
K \uparrow\downarrow r & & K \uparrow\downarrow r \\
\mathbf{M}_*^{\mathcal{C}^{op}} & \underset{F^*}{\overset{F_!'}{\rightleftarrows}} & \mathbf{M}_*^{\mathcal{D}^{op}}
\end{array}
$$

commutative in the obvious sense. Since K is full and faithful, one has $F_!' = rF_!K$.

(f) Let $F : \mathbf{M}_1 \to \mathbf{M}_2$ be a functor which preserves the terminal object. Then F induces a functor $F : \mathbf{M}_{1\,*}^{\mathcal{C}^{op}} \to \mathbf{M}_{2\,*}^{\mathcal{C}^{op}}$. If $F : \mathbf{M}_1 \to \mathbf{M}_2$ is full and faithful, then so is F.

(g) (Change of base category) Let $F : \mathbf{M}_1 \rightleftarrows \mathbf{M}_2 : G$ be an adjoint pair. The induced functor $G : \mathbf{M}_{2\,*}^{\mathcal{C}^{op}} \to \mathbf{M}_{1\,*}^{\mathcal{C}^{op}}$ has a left adjoint F' constructed in such a way that it makes the diagram

$$
\begin{array}{ccc}
\mathbf{M}_1^{\mathcal{C}^{op}} & \underset{G}{\overset{F}{\rightleftarrows}} & \mathbf{M}_2^{\mathcal{C}^{op}} \\
K_1 \uparrow\downarrow r_1 & & K_2 \uparrow\downarrow r_2 \\
\mathbf{M}_{1\,*}^{\mathcal{C}^{op}} & \underset{G}{\overset{F'}{\rightleftarrows}} & \mathbf{M}_{2\,*}^{\mathcal{C}^{op}}
\end{array}
$$

commutative in the obvious sense. Since K_1 is full and faithful, one has $F' = r_2FK_1$. If $F* \cong *$, then $F'* \cong *$.

2. The Reedy and projective model structures for Segal $\mathbf{M}$-categories with fixed set of objects

2.1. The Reedy model structure. Recall from definition 1.1 the functor K.

Proposition 2.1. (compare with [11, 11.7 and 12.3.1]) *Let $\mathbf{M}$ be a model category and $\mathcal{C}$ a small Reedy category. Let $\mathbf{M}^{\mathcal{C}^{op}}$ have the Reedy model structure. Then the category $\mathbf{M}_*^{\mathcal{C}^{op}}$ admits a model category structure with the classes of weak equivalences, cofibrations and fibrations defined via the functor K. $\mathbf{M}_*^{\mathcal{C}^{op}}$ is cofibrantly generated if $\mathbf{M}^{\mathcal{C}^{op}}$ is. $\mathbf{M}_*^{\mathcal{C}^{op}}$ is left proper if $\mathbf{M}$ is.*

Proof. The lifting axiom of a model category is clear. The factorization axiom is shown inductively on the degree of the objects of $\mathcal{C}$, exactly as in [6, 15.3.16]. The only difference with *loc. cit.* is in degree zero, when we choose $\mathbf{Z}_\alpha = *$, for every object α of $\mathcal{C}$ of degree zero. Suppose now that $\mathbf{M}^{\mathcal{C}^{op}}$ is cofibrantly generated. Let $\mathbb{I}$ and $\mathbb{J}$ be generating sets of cofibrations and trivial cofibrations, respectively. By 1.2(a) the sets $r(\mathbb{I})$ and $r(\mathbb{J})$ permit the small object argument. They can be chosen to be generating sets of cofibrations and trivial cofibrations for the model category $\mathbf{M}_*^{\mathcal{C}^{op}}$. Left properness is straightforward. $\qquad\square$

Proposition 2.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between Reedy categories which preserves the 0-filtration. Let $\mathbf{M}$ be a model category. Suppose that the induced adjoint pair $F_! : \mathbf{M}^{\mathcal{C}^{op}} \rightleftarrows \mathbf{M}^{\mathcal{D}^{op}} : F^*$ is a Quillen pair. Then the induced adjoint pair $F_!' : \mathbf{M}_*^{\mathcal{C}^{op}} \rightleftarrows \mathbf{M}_*^{\mathcal{D}^{op}} : F^*$ (1.2(e)) is a Quillen pair.*

We recall from [1, Definition 3.16.1] that a functor between Reedy categories is a **morphism** if it preserves the inverse and direct subcategories.

Let us fix a small Reedy category $\mathcal{C}$ and a degree preserving morphism $p : \mathcal{C} \to \Delta$ which is a fibration. We denote the fibre category of p over $[n] \in \Delta$ by $\mathcal{C}_n$, and the natural functor $\mathcal{C}_n \to \mathcal{C}$ by σ_n. One has $\mathcal{C}_0 = F^0\mathcal{C}$.

For each $n \geq 1$, let $\alpha^k : [1] \to [n]$ be the map in Δ defined as $\alpha^k(0) = k$ and $\alpha^k(1) = k + 1$, where $0 \leq k < n$. If $n \geq 2$, one has $\alpha^k d^0 = \alpha^{k+1} d^1$, for $0 \leq k \leq n - 2$. For each $c \in \mathcal{C}$ of degree $n \geq 1$, let $(\alpha^k)^*c \to c$ be a cartesian lifting of α^k, $0 \leq k < n$, and $(d^i)^*(\alpha^k)^*c \to (\alpha^k)^*c$ be a cartesian lifting of $d^i : [0] \to [1]$, $0 \leq i \leq 1$. We obtain a commutative diagram in $\mathcal{C}$

$$(d^0)^*(\alpha^k)^*c = (d^1)^*(\alpha^{k+1})^*c$$

$$(\alpha^k)^*c \qquad\qquad (\alpha^{k+1})^*c$$

$$c$$

Let now $\mathbf{M}$ be a left proper, combinatorial model category. For $c \in \mathcal{C}$ we denote by ev_c the evaluation at c functor $\mathbf{M}^{\mathcal{C}^{op}} \to \mathbf{M}$. ev_c has a left adjoint F_c which sends $A \in \mathbf{M}$ to

$$F_c^A(c') = \coprod_{\mathcal{C}(c',c)} A$$

Let W be the set considered in [5, Proposition A.5]. We denote by $\mathfrak{S}$ the set

$$\{rF_{(\alpha^0)^*c}^A \underset{rF_{(d^0)^*(\alpha^0)^*c}^A}{\cup} \cdots \underset{rF_{(d^0)^*(\alpha^{n-2})^*c}^A}{\cup} rF_{(\alpha^{n-1})^*c}^A \to rF_c^A\}_{\{c, deg(c) \geq 1\} \times \{A \in W\}}$$

where the map is induced by commutative diagrams as above.

Theorem 2.3. *Let $\mathbf{M}$ be a left proper, combinatorial model category. The category $\mathbf{M}_*^{\mathcal{C}^{op}}$ admits a left proper, combinatorial model category structure with the cofibrations of $\mathbf{M}_*^{\mathcal{C}^{op}}$ as cofibrations and the $\mathfrak{S}$-local equivalences as weak equivalences. We denote this model structure by $\mathbf{M}_{*,S}^{\mathcal{C}^{op}}$, where "S" stands for Segal. If $\mathbf{X}$ is an object of $\mathbf{M}_*^{\mathcal{C}^{op}}$, then $\mathbf{X}$ is fibrant in $\mathbf{M}_{*,S}^{\mathcal{C}^{op}}$ if and only if $\mathbf{X}$ is fibrant in $\mathbf{M}_*^{\mathcal{C}^{op}}$ and for every object c of $\mathcal{C}$ with $deg(c) \geq 1$, the map*

$$\mathbf{X}(c) \to \mathbf{X}((\alpha^0)^*c) \times \cdots \times \mathbf{X}((\alpha^{deg(c)-1})^*c)$$

is a weak equivalence of $\mathbf{M}$.

Proof. The model structure exists by Smith's theorem [5] applied to the model category $\mathbf{M}_*^{\mathcal{C}^{op}}$ from proposition 2.1 and the set $\mathfrak{S}$. It remains to show that the $\mathfrak{S}$-local objects are the ones mentioned in the statement of the theorem. The proof is the same as the proof of [5, Theorem 5.2(c)]. To begin with, notice that to give a map

$$rF_{(\alpha^0)^*c}^A \underset{rF_{(d^0)^*(\alpha^0)^*c}^A}{\cup} \cdots \underset{rF_{(d^0)^*(\alpha^{deg(c)-2})^*c}^A}{\cup} rF_{(\alpha^{deg(c)-1})^*c}^A \to \mathbf{X}$$

where $deg(c) \geq 1$ and $A \in W$, is to give a map $A \to \mathbf{X}((\alpha^0)^*c) \times \cdots \times \mathbf{X}((\alpha^{deg(c)-1})^*c)$. It follows that

$$rF_{(\alpha^0)^*c}^A \underset{rF_{(d^0)^*(\alpha^0)^*c}^A}{\cup} \cdots \underset{rF_{(d^0)^*(\alpha^{deg(c)-2})^*c}^A}{\cup} rF_{(\alpha^{deg(c)-1})^*c}^A$$

is cofibrant in $\mathbf{M}_*^{\mathcal{C}^{op}}$. Using [6, 17.4.16(2)] and [5, Proposition A.5] we arrive at the desired characterization of $\mathfrak{S}$-local objects. $\qquad\square$

Example 2.4. Let **Cat** be the category of all small categories and **S** the category of simplicial sets. For a small category $\mathcal{C}$, we denote by $y_{\mathcal{C}} : \mathcal{C} \to Set^{\mathcal{C}^{op}}$, or simply y, the Yoneda embedding.

Let $\mathcal{C}$ be a small category and $\Psi : \mathcal{C}^{op} \to \mathbf{Cat}$ a functor. The Grothendieck construction $\int_{\mathcal{C}} (\Psi)$ is the category with objects (c, x) where $c \in Ob\mathcal{C}$ and $x \in Ob\Psi(c)$, and arrows $(c, x) \to (d, y)$ are pairs (u, f) with $u : c \to d$ in $\mathcal{C}$ and $f : x \to \Psi(u)(y)$ in $\Psi(c)$. The projection $\int_{\mathcal{C}} (\Psi) \to \mathcal{C}$ is a fibration.

Let $\mathcal{C}$ be a small Reedy category and let $X \in Set^{\mathcal{C}^{op}}$. The comma category $(y \downarrow X)$ is the Grothendieck construction of the composite functor $\mathcal{C}^{op} \overset{X}{\to} Set \overset{D}{\to} \mathbf{Cat}$, where $D : Set \to \mathbf{Cat}$ is the discrete category functor. $(y \downarrow X)$ becomes a Reedy category and the projection $(y \downarrow X) \to \mathcal{C}$ a degree preserving morphism of Reedy categories.

Let $N : \mathbf{Cat} \to \mathbf{S}$ be the nerve functor. If $\mathcal{C}$ is a small category, we put $\Delta\mathcal{C} = (y \downarrow N(\mathcal{C}))$ and $\Delta^{op}\mathcal{C} = (\Delta\mathcal{C})^{op}$. If $([n], c_0 \to ... \to c_n)$ is an object of $\Delta\mathcal{C}$ with $n \geq 2$, the cartesian lifting of $\alpha^k : [1] \to [n]$ is $([1], c_k \to c_{k+1}) \to ([n], c_0 \to ... \to c_n)$.

Let $\iota : Set \to \mathbf{Cat}$ be the indiscrete/chaotic category functor, right adjoint to the set of objects functor. If S is a set, one has $N(\iota S)_n = \prod_{[0] \to [n]} S$. We put $\Delta S = \Delta\iota S$. If $([n], s_0, ..., s_n)$ is an object of ΔS with $n \geq 2$, the cartesian lifting of $\alpha^k : [1] \to [n]$ is $([1], s_k, s_{k+1}) \to ([n], s_0, ..., s_n)$. In [2, Section 5], J. Bergner has made an early use of the category ΔS, in the same context as ours. The existence of the model structure $\mathbf{M}_{*,S}^{\Delta^{op}S}$ was proved in [11, 12.3.2], using a different method. An object $\mathbf{X}$ of $\mathbf{M}_{*,S}^{\Delta^{op}S}$ is fibrant in $\mathbf{M}_{*,S}^{\Delta^{op}S}$ if and only if $\mathbf{X}$ is fibrant in $\mathbf{M}_*^{\Delta^{op}S}$ and for every object $([n], s_0, ..., s_n)$ of ΔS $(n \geq 1)$, the map

$$\mathbf{X}(([n], s_0, ..., s_n)) \to \mathbf{X}(([1], s_0, s_1)) \times \mathbf{X}(([1], s_1, s_2)) \times ... \times \mathbf{X}(([1], s_{n-1}, s_n))$$

is a weak equivalence of $\mathbf{M}$.

Remark 2.5. One can perform the localization at a different set of maps. Here is an example [3, Section 6]. For each $n \geq 1$, let $\gamma^k : [1] \to [n]$ be the map in Δ defined as $\gamma^k(0) = 0$ and $\gamma^k(1) = k + 1$, where $0 \leq k < n$. If $n \geq 2$, one has $\gamma^k d^1 = \gamma^{k+1} d^1$, for $0 \leq k \leq n - 2$. For each $c \in \mathcal{C}$ of degree $n \geq 1$, let $(\gamma^k)^* c \to c$ be a cartesian lifting of γ^k, $0 \leq k < n$, and $(d^1)^*(\alpha^k)^* c \to (\alpha^k)^* c$ be a cartesian lifting of $d^1 : [0] \to [1]$. We obtain a commutative diagram in $\mathcal{C}$

$$(d^1)^*(\gamma^k)^* c = (d^1)^*(\gamma^{k+1})^* c$$

$$(\gamma^k)^* c \qquad\qquad (\gamma^{k+1})^* c$$

$$c$$

and the set of maps

$$\{ rF_{(\gamma^0)^*c}^A \underset{rF_{(d^1)^*(\gamma^0)^*c}^A}{\cup} ... \underset{rF_{(d^1)^*(\gamma^0)^*c}^A}{\cup} rF_{(\gamma^{n-1})^*c}^A \to rF_c^A \}_{\{c, deg(c) \geq 1\} \times \{A \in W\}}$$

We shall study now the behaviour of the model category $\mathbf{M}_{*,S}^{\mathcal{C}^{op}}$ under change of diagram and base category.

Proposition 2.6. *Let $p : \mathcal{C} \to \Delta$ and $q : \mathcal{D} \to \Delta$ be degree preserving morphisms of Reedy categories which are fibrations. Let $F : \mathcal{C} \to \mathcal{D}$ be a degree preserving morphism of Reedy categories such that F is a fibred functor. Let $\mathbf{M}$ be a left proper, combinatorial model category. Suppose that the induced adjoint pair $F_! : \mathbf{M}^{\mathcal{C}^{op}} \rightleftarrows \mathbf{M}^{\mathcal{D}^{op}} : F^*$ is a Quillen pair. Then the induced adjoint pair $F_!' : \mathbf{M}_{*,S}^{\mathcal{C}^{op}} \rightleftarrows \mathbf{M}_{*,S}^{\mathcal{D}^{op}} : F^*$ (1.2(e)) is a Quillen pair.*

Proof. From proposition 2.2 it suffices [6, 8.5.4(3)] to show that F^* preserves fibrations between fibrant objects. But this is clear from the assumptions on F. $\qquad\square$

We recall [1, Definition 3.16.3] that a morphism $f : \mathcal{C} \to \mathcal{D}$ of Reedy categories is a **right fibration** if for every model category $\mathbf{M}$, the adjoint pair $f_! : \mathbf{M}^{\mathcal{C}^{op}} \rightleftarrows \mathbf{M}^{\mathcal{D}^{op}} : f^*$ is a Quillen pair.

Lemma 2.7. *For every map $f : X \to Y$ of simplicial sets, $f : (y \downarrow X) \to (y \downarrow Y)$ is a right fibration.*

Proof. It suffices to prove that $\partial(((([n], y) \downarrow \overrightarrow{f}) \downarrow \lambda)$ is empty or connected, where $\lambda : ([n], y) \to ([m], f(x))$ and λ is a monomorphism. A commutative diagram

$$
\begin{array}{ccc}
 & ([n], y) & \\
\alpha \swarrow & \downarrow \lambda & \searrow \beta \\
([p], f(a)) & & ([q], f(b)) \\
\gamma \searrow & \downarrow & \swarrow \delta \\
 & ([m], f(x)) &
\end{array}
$$

in which $\alpha, \beta, \delta, \gamma$ are monomorphisms can be completed to a commutative diagram

$$
\begin{array}{ccccc}
 & & ([n], y) & & \\
 & \alpha \swarrow & \downarrow \eta & \searrow \beta & \\
([p], f(a)) & \xleftarrow{\theta} & ([r], f(u^*x)) & \xrightarrow{\varepsilon} & ([q], f(b)) \\
 & \gamma \searrow & \downarrow u & \swarrow \delta & \\
 & & ([m], f(x)) & &
\end{array}
$$

in which $\theta, \varepsilon, \eta, u$ are monomorphisms and $\lambda = u\eta$. $\square$

Corollary 2.8. *For any map $f : X \to Y$ of simplicial sets and any model category* $\mathbf{M}$*, the adjoint pair*

$$(y \downarrow f)_! : \mathbf{M}^{(y \downarrow X)^{op}} \rightleftarrows \mathbf{M}^{(y \downarrow Y)^{op}} : (y \downarrow f)^*$$

is a Quillen pair. In particular, if $f : S \to T$ is a function, the adjoint pair

$$f'_! : \mathbf{M}_*^{\Delta^{op}S} \rightleftarrows \mathbf{M}_*^{\Delta^{op}T} : f^*$$

is a Quillen pair.

Corollary 2.9. *Let* $\mathbf{M}$ *be a left proper, combinatorial model category. If $f : S \to T$ is a function, then the adjoint pair $f'_! : \mathbf{M}_{*,S}^{\Delta^{op}S} \rightleftarrows \mathbf{M}_{*,S}^{\Delta^{op}T} : f^*$ is a Quillen pair.*

Proof. Apply corollary 2.8 and proposition 2.6. $\square$

Proposition 2.10. (compare with [11, 12.6.2 and 14.7.3]) *Let $F : \mathbf{M}_1 \rightleftarrows \mathbf{M}_2 : G$ be a Quillen pair between left proper, combintorial model categories. Then the induced adjoint pairs $F' : \mathbf{M}_{1*}^{\mathcal{C}^{op}} \rightleftarrows \mathbf{M}_{2*}^{\mathcal{C}^{op}} : G$ and $F' : \mathbf{M}_{1*,S}^{\mathcal{C}^{op}} \rightleftarrows \mathbf{M}_{2*,S}^{\mathcal{C}^{op}} : G$ (1.2(g)) are Quillen pairs. If $F' : \mathbf{M}_{1*}^{\mathcal{C}^{op}} \to \mathbf{M}_{2*}^{\mathcal{C}^{op}}$ preserves weak equivalences then so does $F' : \mathbf{M}_{1*,S}^{\mathcal{C}^{op}} \to \mathbf{M}_{2*,S}^{\mathcal{C}^{op}}$.*

Proof. It is clear that the first pair is a Quillen pair. For the second it suffices [6, 8.5.4(3)] to show that G preserves fibrations between fibrant objects. But this is clear since G preserves weak equivalences between fibrant objects. The rest is a consequence of general facts about left Bousfield localizations [6, Chapter 3]. $\square$

Let us illustrate the last part of the previous proposition. If $\mathcal{C}$ is an elegant Reedy category [4, Definition 3.5] and the cofibrations of $\mathbf{M}_1$ are the monomorphisms, then $F' : \mathbf{M}_{1*}^{\mathcal{C}^{op}} \to \mathbf{M}_{2*}^{\mathcal{C}^{op}}$ preserves weak equivalences.

2.2. **The projective model structure.**

Proposition 2.11. [11, 11.4.2] *Let $\mathbf{M}$ be a cofibrantly generated model category. Let $\mathcal{C}$ be a small Reedy category and S a set. The category $\mathbf{M}_*^{\mathcal{C}^{op}}$ admits a cofibrantly generated model category structure obtained by transfer from the projective model structure on $\mathbf{M}^{\mathcal{C}^{op}}$ via the adjunction (r, K) (1.2(a)). We denote this model structure by $\mathbf{M}_{*,p}^{\mathcal{C}^{op}}$. $\mathbf{M}_{*,p}^{\mathcal{C}^{op}}$ is left proper if $\mathbf{M}$ is.*

It follows from the proof of [11, 11.4.2] that a cofibrant object of $\mathbf{M}_{*,p}^{\mathcal{C}^{op}}$ is objectwise cofibrant. We also observe that the identity pair $Id : \mathbf{M}_{*,p}^{\mathcal{C}^{op}} \rightleftarrows \mathbf{M}_*^{\mathcal{C}^{op}} : Id$ is a Quillen pair.

For the next result, recall from 2.1 the set $\mathfrak{S}$.

Theorem 2.12. (compare with [11, 12.1.1]) *Let* $\mathbf{M}$ *be a left proper, combinatorial model category. Let* $\mathcal{C}$ *be a small Reedy category and* $p : \mathcal{C} \to \Delta$ *a degree preserving morphism which is a fibration. The category* $\mathbf{M}_*^{\mathcal{C}^{op}}$ *admits a left proper, combinatorial model category structure with the cofibrations of* $\mathbf{M}_{*,p}^{\mathcal{C}^{op}}$ *as cofibrations and the* $\mathfrak{S}$*-local equivalences as weak equivalences. We denote this model structure by* $\mathbf{M}_{*,p,S}^{\mathcal{C}^{op}}$, *where "S" stands for Segal. If* $\mathbf{X}$ *is an object of* $\mathbf{M}_*^{\mathcal{C}^{op}}$, *then* $\mathbf{X}$ *is fibrant in* $\mathbf{M}_{*,p,S}^{\mathcal{C}^{op}}$ *if and only if* $\mathbf{X}$ *is fibrant in* $\mathbf{M}_{*,p}^{\mathcal{C}^{op}}$ *and for every object* c *of* $\mathcal{C}$ *with* $deg(c) \geq 1$, *the map*

$$\mathbf{X}(c) \to \mathbf{X}((\alpha^0)^*c) \times ... \times \mathbf{X}((\alpha^{deg(c)-1})^*c)$$

is a weak equivalence of $\mathbf{M}$.

Proof. The proof is the same as for theorem 2.3, using proposition 2.11 instead of proposition 2.1. $\square$

Theorem 2.13. (compare with [11, 12.3.2]) *The weak equivalences of* $\mathbf{M}_{*,p,S}^{\mathcal{C}^{op}}$ *and* $\mathbf{M}_{*,S}^{\mathcal{C}^{op}}$ *are the same.*

Proof. Apply lemma 7.2 and the previous considerations. $\square$

3. SEGAL $\mathbf{M}$-CATEGORIES AND $\mathbf{M}$-CATEGORIES (WITH FIXED SET OF OBJECTS)

Let $\mathbf{M}$ be a cocomplete cartesian closed category. Let S be a set. We denote by $\mathbf{M}\text{-}\mathbf{Cat}(S)$ the category of small $\mathbf{M}$-categories with fixed set of objects S. Recall [11],[7] that there is a functor $N : \mathbf{M}\text{-}\mathbf{Cat}(S) \to \mathbf{M}_*^{\Delta^{op}S}$ constructed as

$$N\mathcal{A}(([n], s_0, ..., s_n)) = \begin{cases} *, & \text{if } n = 0 \\ \underline{\mathcal{A}}(s_0, ..., s_n), & \text{otherwise} \end{cases}$$

where $\underline{\mathcal{A}}(s_0, ..., s_n) = \mathcal{A}(s_0, s_1) \times ... \times \mathcal{A}(s_{n-1}, s_n)$. For example, if A is a monoid in $\mathbf{M}$, then NA is the simplicial bar construction of the (trivially) augmented monoid A. N is full and faithful. $\mathbf{X} \in \mathbf{M}_*^{\Delta^{op}S}$ is in the essential image of N if and only if for every object $([n], s_0, ..., s_n)$ of ΔS, the canonical map

$$\mathbf{X}(([n], s_0, ..., s_n)) \to \mathbf{X}(([1], s_0, s_1)) \times ... \times \mathbf{X}(([1], s_{n-1}, s_n))$$

is an isomorphism. N has a left adjoint L constructed explicitly in [7, 2.2]: to every pair x, y of elements of S a certain category $\mathcal{J}_{x,y}(S)$ is associated, and $L\mathbf{X}(x, y)$ is the colimit of a certain functor $H_{x,y}^{\mathbf{X}} : \mathcal{J}_{x,y}(S) \to \mathbf{M}$ associated to $\mathbf{X}$.

For the next result, we regard $\mathbf{M}\text{-}\mathbf{Cat}(S)$ as having the standard [10] model structure.

Theorem 3.1. (J. Lurie) *Let* $\mathbf{M}$ *be a left proper, combinatorial cartesian model category with cofibrant unit, having a set of generating cofibrations with cofibrant domains and satisfying the monoid axiom of* [9]. *Let* S *be a set. Then the adjoint pair*

$$L : \mathbf{M}_{*,p,S}^{\Delta^{op}S} \rightleftarrows \mathbf{M}\text{-}\mathbf{Cat}(S) : N$$

is a Quillen equivalence.

The above theorem was proved by J. Bergner [2] in the case $\mathbf{M}=\mathbf{S}$, using algebraic theories. It was also proved in [7, Theorem 2.2.16], not quite in this form, using a different method and under the assumption that all objects of $\mathbf{M}$ are cofibrant. However, a close analysis of the proof of [7, Theorem 2.2.16] reveals that this assumption is superfluous. To make things clear we give below Lurie's proof, stripped to the essentials and with some changes. At the heart of it is the following technical result.

Proposition 3.2. [7, Lemma 2.2.15] *Let* $\mathbf{M}$ *be a left proper, combinatorial, cartesian simplicial model category. Let* S *be a set. Let* $\mathbf{X} \in \mathbf{M}_*^{\Delta^{op}S}$ *be cofibrant in* $\mathbf{M}_{*,p}^{\Delta^{op}S}$ *and such that for every object* $([n], s_0, ..., s_n)$ *of* ΔS, *the canonical map*

$$\mathbf{X}(([n], s_0, ..., s_n)) \to \mathbf{X}(([1], s_0, s_1)) \times ... \times \mathbf{X}(([1], s_{n-1}, s_n))$$

is a weak equivalence. Then for all pairs x, y *of elements of* S, *the canonical map* $\mathbf{X}(([1], x, y)) \to NL\mathbf{X}(([1], x, y))$ *is a weak equivalence of* $\mathbf{M}$.

Proof. We use the notations of the proof of *loc. cit.*. Thus, we have a commutative diagram

$$
\begin{array}{ccc}
\mathbf{X}(([1],x,y)) & \longrightarrow & NL\mathbf{X}(([1],x,y)) \\
\cong \big\uparrow & & \big\uparrow \cong \\
colim_{\mathcal{J}'_{x,y}(S)} H & \longrightarrow & colim_{\mathcal{J}_{x,y}(S)} H^{\mathbf{X}}_{x,y} \\
\big\uparrow & & \big\uparrow \\
hocolim_{\mathcal{J}'_{x,y}(S)} H & \longrightarrow & hocolim_{\mathcal{J}_{x,y}(S)} H^{\mathbf{X}}_{x,y}
\end{array}
$$

The right vertical map is a weak equivalence by sublemma 3.3 since $H^{\mathbf{X}}_{x,y}$ is cofibrant [7, Proposition 2.2.6]. The left vertical map is a weak equivalence since $\mathcal{J}'_{x,y}(S)$ has terminal object. We prove that the bottom horizontal map is a weak equivalence. Let i be the inclusion $\mathcal{J}'_{x,y}(S) \subset \mathcal{J}_{x,y}(S)$. There are a functor $R : \mathcal{J}_{x,y}(S) \to \mathcal{J}'_{x,y}(S)$ and a natural transformation $\alpha : iR \Rightarrow Id$. ¿From the second assumption on $\mathbf{X}$ it follows that for every object σ of $\mathcal{J}_{x,y}(S)$, the map $H^{\mathbf{X}}_{x,y}(\alpha_\sigma)$ is a weak equivalence. We are then in the situation of sublemma 3.4 with $G = i$, $F = R$ and $Ri = Id$. $\qquad\square$

For the next two (standard) results, $hocolim_I \mathbf{X}$ stands for the homotopy colimit of $\mathbf{X}$, as in [6, 18.1.2].

Sublemma 3.3. *Let $\mathbf{M}$ be a cofibrantly generated simplicial model category and I a small category. Then for every cofibrant object $\mathbf{X} \in \mathbf{M}^I$, the natural map*

$$hocolim_I \mathbf{X} \to colim_I \mathbf{X}$$

is a weak equivalence.

Sublemma 3.4. *Let $\mathbf{M}$ be a simplicial model category I and J two small categories. Suppose that are functors $G : I \to J$, and $F : J \to I$ together with natural transformations $\alpha : GF \Rightarrow Id_J$ and $\beta : FG \Rightarrow Id_I$. Let $\mathbf{X} : J \to \mathbf{M}$ take cofibrant values and be such that*
(a) $\mathbf{X}(\alpha_j) : \mathbf{X}GF(j) \to \mathbf{X}(j)$ is a weak equivalence for all $j \in J$, and
(b) $\mathbf{X}(G(\beta_i)) : \mathbf{X}GFG(i) \to \mathbf{X}G(i)$ is a weak equivalence for all $i \in I$.
Then the map

$$hocolim_I \mathbf{X}G \to hocolim_J \mathbf{X}$$

is a weak equivalence.

We begin now the proof of theorem 3.1. It is clear that (L, N) is a Quillen pair and that N preserves and reflects weak equivalences between fibrant objects. We prove that the total left derived functor of L is full and faithful. This amounts to showing that for every $\mathbf{X} \in \mathbf{M}^{\Delta^{op}S}_*$ which is cofibrant-fibrant in $\mathbf{M}^{\Delta^{op}S}_{*,p,S}$, and for some fibrant approximation $L\mathbf{X} \to \hat{F}L\mathbf{X}$ to $L\mathbf{X}$, the map $\mathbf{X} \to NL\mathbf{X} \to N\hat{F}L\mathbf{X}$ is a weak equivalence in $\mathbf{M}^{\Delta^{op}S}_{*,p,S}$. We factor the map $L\mathbf{X} \to *$ as a trivial cofibration $L\mathbf{X} \to \hat{F}L\mathbf{X}$ followed by a fibration $\hat{F}L\mathbf{X} \to *$. We shall prove that $\mathbf{X} \to NL\mathbf{X} \to N\hat{F}L\mathbf{X}$ is a weak equivalence in $\mathbf{M}^{\Delta^{op}S}_{*,p}$. Since $L\mathbf{X}$ is cofibrant, the map $NL\mathbf{X} \to N\hat{F}L\mathbf{X}$ is a weak equivalence in $\mathbf{M}^{\Delta^{op}S}_{*,p}$. Since $\mathbf{X}$ is fibrant, to prove that $\mathbf{X} \to NL\mathbf{X}$ is a weak equivalence it suffices to prove that for every pair x, y of elements of S, the map $\mathbf{X}(([1],x,y)) \to L\mathbf{X}(x,y)$ is a weak equivalence.

Let $L_\mathfrak{S}\mathbf{M}^{\Delta^{op}}$ be the model category considered in the proof of proposition 7.1. Since $cst : \mathbf{M} \to L_\mathfrak{S}\mathbf{M}^{\Delta^{op}}$ reflects weak equivalences between cofibrant objects, it suffices to show that $cst\mathbf{X}(([1],x,y)) \to cstL\mathbf{X}(x,y)$ is a weak equivalence.

Consider the diagram

$$
\begin{array}{ccc}
\mathbf{X} \in \mathbf{M}^{\Delta^{op}S}_* & \underset{N}{\overset{L}{\rightleftarrows}} & \mathbf{M}\text{-}\mathbf{Cat}(S) \\
ev_0 \big\uparrow\big\downarrow cst' & & ev_0 \big\uparrow\big\downarrow cst \\
(L_\mathfrak{S}\mathbf{M}^{\Delta^{op}})^{\Delta^{op}S}_* & \underset{N}{\overset{L}{\rightleftarrows}} & L_\mathfrak{S}\mathbf{M}^{\Delta^{op}}\text{-}\mathbf{Cat}(S)
\end{array}
$$

One has $ev_0 N = N ev_0$, so $Lcst' \cong cstL$. We will prove that the object $\mathbf{Z} = cst'\mathbf{X}$ satisfies the assumptions of proposition 3.2. For this, it suffices to prove that for every object $([n], s_0, ..., s_n)$ of ΔS, the canonical map

$$\mathbf{Z}(([n], s_0, ..., s_n)) \to \mathbf{Z}(([1], s_0, s_1)) \times ... \times \mathbf{Z}(([1], s_{n-1}, s_n))$$

is a weak equivalence in $\mathbf{M}^{\Delta^{op}}$. Under the isomorphism $(\mathbf{M}^{\Delta^{op}})_*^{\Delta^{op}S} \cong (\mathbf{M}_*^{\Delta^{op}S})^{\Delta^{op}}$ (1.2(d)) $\mathbf{Z}$ corresponds to $cst\mathbf{X}$, and then the fact that $\mathbf{X}$ is fibrant implies that the required map is a weak equivalence. Thus, by proposition 3.2 the map $\mathbf{Z}(([1], x, y)) \to L\mathbf{Z}(x, y)$ is a weak equivalence. But $L\mathbf{Z}(x, y) \cong cstL\mathbf{X}(x, y)$ and $\mathbf{Z}(([1], x, y)) \cong cst\mathbf{X}(([1], x, y))$. The proof of theorem 3.1 is complete.

4. The category of pre-$\mathbf{M}$-categories

Recall from example 2.4 the category ΔS. Let $\mathbf{M}$ be a category. A function $u : S \to T$ induces a functor $u^* : \mathbf{M}^{\Delta^{op}T} \to \mathbf{M}^{\Delta^{op}S}$, which has a left adjoint $u_!$ provided that $\mathbf{M}$ is cocomplete, and a right adjoint u_* provided that $\mathbf{M}$ is complete.

Definition 4.1. *We define a category $\mathbf{M}^{\Delta^{op}Set}$ as follows. The objects of $\mathbf{M}^{\Delta^{op}Set}$ are pairs $(S, \mathbf{X})$, where S is a set and $\mathbf{X} \in \mathbf{M}^{\Delta^{op}S}$. An arrow $(S, \mathbf{X}) \to (T, \mathbf{Y})$ is a pair (u, f), where $u : S \to T$ is a function and $f : \mathbf{X} \Rightarrow u^*\mathbf{Y}$ is a natural tranformation.*

Suppose that $\mathbf{M}$ is suitably complete and cocomplete. The terminal object of $\mathbf{M}^{\Delta^{op}Set}$ is $(*, *)$. The functor $Ob : \mathbf{M}^{\Delta^{op}Set} \to Set$ defined as $Ob((S, \mathbf{X})) = S$ has a left adjoint D given by $DS = (S, \emptyset)$, and a right adjoint ι given by $\iota S = (S, *)$. D and ι are full and faithful. The functor Ob is a cloven Grothendieck bifibration. An arrow $(u, f) : (S, \mathbf{X}) \to (T, \mathbf{Y})$ is cartesian if and only if f is an isomorphism. The fibre category of Ob over a set S is $\mathbf{M}^{\Delta^{op}S}$.

Suppose that $\mathbf{M}$ is a closed category. Write Y^X for the internal hom of two objects X, Y of $\mathbf{M}$. Then for every set S, $\mathbf{M}^{\Delta^{op}S}$ is tensored and cotensored over $\mathbf{M}$ (1.2(c)). For every function $u : S \to T$, $\mathbf{X} \in \mathbf{M}^{\Delta^{op}T}$ and $A \in \mathbf{M}$ we have the formula $u^*(\mathbf{X}^A) = (u^*\mathbf{X})^A$, hence $\mathbf{M}^{\Delta^{op}Set}$ is tensored and cotensored over $\mathbf{M}$, with tensor $A \star (S, \mathbf{X}) = (S, A \star \mathbf{X})$ and cotensor $(S, \mathbf{X})^A = (S, \mathbf{X}^A)$.

4.1. **The category of pre-M-categories.** Let S be a set. Recall from section 1 the inclusion $\sigma_0 : S \subset \Delta S$.

Definition 4.2. *Let $\mathbf{M}$ be a category with terminal object. The category $\mathbf{M}_*^{\Delta^{op}Set}$ is the full subcategory of $\mathbf{M}^{\Delta^{op}Set}$ on objects $(S, \mathbf{X})$ such that $\sigma_0^*\mathbf{X} = *$. The objects of $\mathbf{M}_*^{\Delta^{op}Set}$ are referred to as* **pre-M-categories***, and the arrows of $\mathbf{M}_*^{\Delta^{op}Set}$ as* **pre-M-functors***. If $(S, \mathbf{X})$ is a pre-$\mathbf{M}$-category, then S is its* **set of objects***.*

The category $\mathbf{M}_*^{\Delta^{op}Set}$ is denoted by $\mathcal{PC}(\mathcal{M})$ in [11]. We let K be the inclusion functor $\mathbf{M}_*^{\Delta^{op}Set} \subset \mathbf{M}^{\Delta^{op}Set}$. K has a left adjoint r calculated (1.2(a)) as $r((S, \mathbf{X})) = (S, r\mathbf{X})$. We denote by Ob the composite ObK, and by D the composite rD. One has

$$DS(([n], s_0, ..., s_n)) = \begin{cases} *, & \text{if } s_0 = ... = s_n \\ \emptyset, & \text{otherwise} \end{cases}$$

Note that $\iota : Set \to \mathbf{M}^{\Delta^{op}Set}$ takes values in $\mathbf{M}_*^{\Delta^{op}Set}$, D and ι are full and faithful, D is left adjoint to Ob and ι is right adjoint to Ob. The functor Ob is a cloven Grothendieck bifibration. An arrow $(u, f) : (S, \mathbf{X}) \to (T, \mathbf{Y})$ is cartesian if and only if f is an isomorphism [11, 10.3]. The fibre category of Ob over a set S is $\mathbf{M}_*^{\Delta^{op}S}$. K becomes a cartesian functor. To give a map $(*, *) \to (S, \mathbf{X})$ in $\mathbf{M}_*^{\Delta^{op}Set}$ is to give an object of $\mathbf{X}$.

We shall compute $Set_*^{\Delta^{op}Set}$. Recall that $\mathbf{S}$ denotes the category of simplicial sets. Recall also that for every small category $\mathcal{C}$ and every $X \in Set^{\mathcal{C}^{op}}$ there is an equivalence of categories

$$(Set^{\mathcal{C}^{op}} \downarrow X) \overset{\cong}{\Rightarrow} Set^{(y \downarrow X)^{op}}$$

In particular, for every set U there is an equivalence of categories

$$(\mathbf{S} \downarrow N\iota U) \overset{\cong}{\Rightarrow} Set^{\Delta^{op}U}$$

Under the above equivalence $\mathbf{S}_U$ corresponds to $Set_*^{\Delta^{op}U}$. It follows that $\mathbf{S}$ is equivalent to $Set_*^{\Delta^{op}Set}$.

More generally, we compute $(Set^{\mathcal{C}})_*^{\Delta^{op}Set}$. Let U be a set. From the previous calculation and 1.2(d) it follows that $(Set^{\mathcal{C}})_*^{\Delta^{op}U}$ is equivalent to the full subcategory of $Set^{\Delta^{op} \times \mathcal{C}}$ consisting of those objects $X : \Delta^{op} \times \mathcal{C} \to Set$ such that $X([0], c) = U$ for all $c \in \mathcal{C}$. If $\mathcal{C}$ is connected, $(Set^{\mathcal{C}})_*^{\Delta^{op}Set}$ is equivalent with the full subcategory of $Set^{\Delta^{op} \times \mathcal{C}}$ consisting of those objects which take every map in $\{[0]\} \times \mathcal{C}$ to an isomorphism. For example, $\mathbf{S}_*^{\Delta^{op}Set}$ is the full subcategory of $\mathbf{S}^{\Delta^{op}}$ on those bisimplicial sets X with X_0 a constant/discrete simplicial set. For a general $\mathcal{C}$, let $\mathcal{D}$

be the pushout

$$
\begin{array}{ccc}
\{[0]\} \times \mathcal{C} & \longrightarrow & \Delta^{op} \times \mathcal{C} \\
\downarrow & & \downarrow \\
* & \longrightarrow & \mathcal{D}
\end{array}
$$

Then $(Set^{\mathcal{C}})_*^{\Delta^{op}Set}$ is equivalent to $Set^{\mathcal{D}}$.

Suppose that $\mathbf{M}$ is a closed category. Then $\mathbf{M}_*^{\Delta^{op}Set}$ is cotensored over $\mathbf{M}$, with cotensor $(S, \mathbf{X})^A = (S, (K\mathbf{X})^A)$. Since $K((S, \mathbf{X})^A) = (K(S, \mathbf{X}))^A$, it follows that $\mathbf{M}_*^{\Delta^{op}Set}$ is tensored over $\mathbf{M}$, with tensor $A \bigstar (S, \mathbf{X}) = r(A \bigstar K(S, \mathbf{X}))$.

Let $(u, f) : (S, \mathbf{X}) \to (T, \mathbf{Y})$ be a map in $\mathbf{M}_*^{\Delta^{op}Set}$. Then (u, f) decomposes (like any map of a bifibration) as $(S, \mathbf{X}) \to (S, u^*\mathbf{Y}) \to (T, \mathbf{Y})$ and as $(S, \mathbf{X}) \to (T, u_!\mathbf{X}) \to (T, \mathbf{Y})$. When u is a monomorphism, $u_!$ has a convenient description [11, 10.3]:

$$
u_!\mathbf{X}(([n], t_0, ..., t_n)) = \begin{cases} \mathbf{X}(([n], s_0, ..., s_n)), & \text{if } t_i = u(s_i), 0 \le i \le n \\ *, & \text{if } t_0 = ... = t_n \in T - S \\ \emptyset, & \text{otherwise} \end{cases}
$$

Lemma 4.3. *Let $(u, f) : (S, \mathbf{X}) \to (T, \mathbf{Y})$ be a map in $\mathbf{M}_*^{\Delta^{op}Set}$. The following are equivalent:*
  (a) *(u, f) is a monomorphism;*
  (b) *u is a monomorphism and $f : \mathbf{X} \to u^*\mathbf{Y}$ is a monomorphism in $\mathbf{M}_*^{\Delta^{op}S}$;*
  (c) *u is a monomorphism and $u_!\mathbf{X} \to \mathbf{Y}$ is a monomorphism in $\mathbf{M}_*^{\Delta^{op}T}$.*

Proof. $(a) \Leftrightarrow (b)$ is a standard argument. $(b) \Leftrightarrow (c)$ by the description of $u_!\mathbf{X}$. $\qquad \square$

4.2. **Relation with enriched categories.** Let $\mathbf{M}$ be a cocomplete cartesian closed category. We denote by $\mathbf{M}$-**Cat** the category of small categories enriched over $\mathbf{M}$. We recall [11],[7] that there is a functor $N : \mathbf{M}$-**Cat** $\to \mathbf{M}_*^{\Delta^{op}Set}$ constructed as follows. If $\mathcal{A}$ is an $\mathbf{M}$-category, $N\mathcal{A} = (Ob\mathcal{A}, \underline{N}\mathcal{A})$, where

$$
\underline{N}\mathcal{A}(([n], s_0, ..., s_n)) = \begin{cases} *, & \text{if } n = 0 \\ \underline{\mathcal{A}}(s_0, ..., s_n), & \text{otherwise} \end{cases}
$$

and $\underline{\mathcal{A}}(s_0, ..., s_n) = \mathcal{A}(s_0, s_1) \times ... \times \mathcal{A}(s_{n-1}, s_n)$. N is full and faithful and cartesian. A pre-$\mathbf{M}$-category $\mathbf{X}$ is in the essential image of N if and only if for every object $([n], s_0, ..., s_n)$ of ΔS, the canonical map

$$
\mathbf{X}(([n], s_0, ..., s_n)) \to \mathbf{X}(([1], s_0, s_1)) \times ... \times \mathbf{X}(([1], s_{n-1}, s_n))
$$

is an isomorphism.

5. Fibred Reedy model structures on pre-$\mathbf{M}$-categories

For the next result, recall from example 2.4 the category ΔS and from proposition 2.1 the model category $\mathbf{M}_*^{\Delta^{op}S}$.

Theorem 5.1. *Let $\mathbf{M}$ be a model category. The category $\mathbf{M}_*^{\Delta^{op}Set}$ admits a model category structure, denoted by $f\mathbf{M}_*^{\Delta^{op}Set}$, in which a map $(u, f) : (S, \mathbf{X}) \to (T, \mathbf{Y})$ is a*
  • *weak equivalence if u is bijective and $f : \mathbf{X} \to u^*\mathbf{Y}$ is a weak equivalence in $\mathbf{M}_*^{\Delta^{op}S}$,*
  • *cofibration if $u_!\mathbf{X} \to \mathbf{Y}$ is a cofibration in $\mathbf{M}_*^{\Delta^{op}T}$,*
  • *fibration if $f : \mathbf{X} \to u^*\mathbf{Y}$ is a fibration in $\mathbf{M}_*^{\Delta^{op}S}$.*

Proof. The functor $Ob : \mathbf{M}_*^{\Delta^{op}Set} \to Set$ is a bifibration (section 4.1). Let Set have the minimal model structure, in which the weak equivalences are the isomorphisms and all maps are cofibrations as well as fibrations. Using corollary 2.8 one can check that the conditions of [12, 2.3, Theorem] are satisfied. $\qquad \square$

Definition 5.2. *Let $\mathbf{M}$ be a left proper, combinatorial model category. A map $(u, f) : (S, \mathbf{X}) \to (T, \mathbf{Y})$ of $\mathbf{M}_*^{\Delta^{op}Set}$ is a*
  (a) **fibred weak equivalence** *if u is bijective and $f : \mathbf{X} \to u^*\mathbf{Y}$ is a weak equivalence in $\mathbf{M}_{*,S}^{\Delta^{op}S}$;*
  (b) **fibred cofibration** *if $u_!\mathbf{X} \to \mathbf{Y}$ is a cofibration in $\mathbf{M}_{*,S}^{\Delta^{op}T}$;*
  (c) **fibred fibration** *if $f : \mathbf{X} \to u^*\mathbf{Y}$ is a fibration in $\mathbf{M}_{*,S}^{\Delta^{op}S}$.*

A map which is both a fibred weak equivalence and fibred cofibration is called isotrivial cofibration in [11, 14.3].

Theorem 5.3. *Let* $\mathbf{M}$ *be a left proper, combinatorial model category. The classes of fibred weak equivalences, fibred cofibrations and fibred fibrations form a model category structure on* $\mathbf{M}_*^{\Delta^{op}Set}$. *We denote this model structure by* $f\mathbf{M}_{*,S}^{\Delta^{op}Set}$. $f\mathbf{M}_{*,S}^{\Delta^{op}Set}$ *is a left Bousfield localization of* $f\mathbf{M}_*^{\Delta^{op}Set}$.

Proof. The functor $Ob : \mathbf{M}_*^{\Delta^{op}Set} \to Set$ is a bifibration (section 4.1). Let Set have the minimal model structure, in which the weak equivalences are the isomorphisms and all maps are cofibrations as well as fibrations. For every set S, we let the category $\mathbf{M}_*^{\Delta^{op}S}$ have the model structure $\mathbf{M}_{*,S}^{\Delta^{op}S}$ of theorem 2.3. Using corollary 2.9 one can check that the conditions of [12, 2.3, Theorem] are satisfied. The rest is clear. $\qquad\square$

5.1. **A weak factorization system.**

Definition 5.4. *Let* $\mathbf{M}$ *be a left proper, combinatorial model category. We say that a map* $(u, f) : (S, \mathbf{X}) \to (T, \mathbf{Y})$ *of* $\mathbf{M}_*^{\Delta^{op}Set}$ *is a*
 (a) **Reedy cofibration** *if* u *is injective and* $u_!\mathbf{X} \to \mathbf{Y}$ *is a cofibration in* $\mathbf{M}_{*,S}^{\Delta^{op}T}$;
 (b) **Reedy trivial fibration** *if* u *is surjective and* $f : \mathbf{X} \to u^*\mathbf{Y}$ *is a trivial fibration in* $\mathbf{M}_{*,S}^{\Delta^{op}S}$.

Proposition 5.5. *Let* $\mathbf{M}$ *be a left proper, combinatorial model category. The pair* (Reedy cofibrations, Reedy trivial fibrations) *is a weak factorization system on* $\mathbf{M}_*^{\Delta^{op}Set}$.

Proof. The functor $Ob : \mathbf{M}_*^{\Delta^{op}Set} \to Set$ is a bifibration. Let Set have the weak factorization system (monomorphisms, epimorphisms) and, for every set S, $\mathbf{M}_{*,S}^{\Delta^{op}S}$ have the weak factorization system (cofibrations, trivial fibrations). Then corollary 2.9 and [12, 2.2] imply the proposition. $\qquad\square$

For every simplicial set X, $(y \downarrow X)$ (example 2.4) is an EZ-Reedy category [4, Definition 4.1]. The next result is then a consequence of lemma 4.3.

Corollary 5.6. (compare with [11, 13.7.2]) *Let* $\mathbf{M}$ *be a combinatorial model category such that the cofibrations of* $\mathbf{M}$ *are the monomorphisms. Then a map of* $\mathbf{M}_*^{\Delta^{op}Set}$ *is a Reedy cofibration if and only if it is a monomorphism.*

We recall [1, Definition 3.16.2] that a morphism $f : \mathcal{C} \to \mathcal{D}$ of Reedy categories is a **left fibration** if for every model category $\mathbf{M}$, the adjoint pair $f^* : \mathbf{M}^{\mathcal{D}^{op}} \rightleftarrows \mathbf{M}^{\mathcal{C}^{op}} : f_*$ is a Quillen pair.

Lemma 5.7. [1, Proof of Theorem 3.51] *Let* $f : \mathcal{C} \to \mathcal{D}$ *be a morphism of Reedy categories. Suppose that* $\mathcal{C}$ *has fibrant constants and every arrow of the inverse subcategory* $\overleftarrow{\mathcal{D}}$ *is an epimorphism. Then* f *is a left fibration.*

Proof. It suffices to prove that $\partial(\alpha \downarrow (\overleftarrow{f \downarrow d}))$ is empty or connected. Let $\alpha = (c, f(c) \to d)$. Since $\partial(c \downarrow \overleftarrow{\mathcal{C}})$ is empty or connected, it all reduces to proving, as in [1, Proof of Theorem 3.51], that the lower triangle in a diagram

commutes. This is where the assumption on $\overleftarrow{\mathcal{D}}$ comes in. $\qquad\square$

Corollary 5.8. *For any map* $f : X \to Y$ *is of simplicial sets and any model category* $\mathbf{M}$, *the adjoint pair*

$$(y \downarrow f)^* : \mathbf{M}^{(y \downarrow Y)^{op}} \rightleftarrows \mathbf{M}^{(y \downarrow X)^{op}} : (y \downarrow f)_*$$

is a Quillen pair.

Lemma 5.9. *Let* $\mathbf{M}$ *be a left proper, combinatorial model category. If* $(u, f) : (S, \mathbf{X}) \to (T, \mathbf{Y})$ *is a Reedy cofibration, then* $f : \mathbf{X} \to u^*\mathbf{Y}$ *is a cofibration in* $\mathbf{M}_{*,S}^{\Delta^{op}S}$.

Proof. Let f_u be the map $u_!\mathbf{X} \to \mathbf{Y}$. By definition, f_u is a cofibration in $\mathbf{M}_*^{\Delta^{op}T}$, hence by corollary 5.8 suitably applied (see 1.2(f)), u^*f_u is a cofibration in $\mathbf{M}_*^{\Delta^{op}S}$. But $u^*f_u \cong f$ since u is a monomorphism. $\qquad\square$

Lemma 5.10. *Let* $\mathbf{M}$ *be a left proper, combinatorial model category. Then fibred weak equivalences are stable under pushout along Reedy cofibrations.*

Proof. Since $f\mathbf{M}_{*,S}^{\Delta^{op}Set}$ is a left Bousfield localization of $f\mathbf{M}_{*}^{\Delta^{op}Set}$, it suffices to prove the lemma for the weak equivalences of $f\mathbf{M}_{*}^{\Delta^{op}Set}$. So let $(v,f) : (R,\mathbf{X}) \to (S,\mathbf{Y})$ be a weak equivalence of $f\mathbf{M}_{*}^{\Delta^{op}Set}$ and $(u,g) : (R,\mathbf{X}) \to (T,\mathbf{Z})$ a Reedy cofibration. Without loss of generality we may assume that v is the identity map of S. The pushout of $(1_S, f)$ along (u,g) is calculated as the pushout

$$
\begin{array}{ccc}
u_! \mathbf{X} & \xrightarrow{u_! f} & u_! \mathbf{Y} \\
\downarrow & & \downarrow \\
\mathbf{Z} & \longrightarrow & \mathbf{P}
\end{array}
$$

in $\mathbf{M}_{*}^{\Delta^{op}T}$. From the explicit description of $u_! \mathbf{X}$ and $u_! \mathbf{Y}$ the map $u_! f$ is a weak equivalence in $\mathbf{M}_{*}^{\Delta^{op}T}$, and the lemma follows since $\mathbf{M}_{*}^{\Delta^{op}T}$ is left proper. $\qquad\square$

6. Fibred projective model structures on pre-$\mathbf{M}$-categories

For the next result, recall from example 2.4 the category ΔS and from proposition 2.11 the model category $\mathbf{M}_{*,p}^{\Delta^{op}S}$.

Theorem 6.1. *Let $\mathbf{M}$ be a cofibrantly generated model category. The category $\mathbf{M}_{*}^{\Delta^{op}Set}$ admits a model category structure, denoted by $f\mathbf{M}_{*,p}^{\Delta^{op}Set}$, in which a map $(u,f) : (S,\mathbf{X}) \to (T,\mathbf{Y})$ is a*

- *weak equivalence if u is bijective and $f : \mathbf{X} \to u^* \mathbf{Y}$ is a weak equivalence in $\mathbf{M}_{*,p}^{\Delta^{op}S}$,*
- *cofibration if $u_! \mathbf{X} \to \mathbf{Y}$ is a cofibration in $\mathbf{M}_{*,p}^{\Delta^{op}T}$,*
- *fibration if $f : \mathbf{X} \to u^* \mathbf{Y}$ is a fibration in $\mathbf{M}_{*,p}^{\Delta^{op}S}$.*

Proof. The proof is the same as for theorem 5.1, using the fact that if $f : S \to T$ is a function, the adjoint pair $f'_! : \mathbf{M}_{*,p}^{\Delta^{op}S} \rightleftarrows \mathbf{M}_{*,p}^{\Delta^{op}T} : f^*$ is a Quillen pair. $\qquad\square$

Definition 6.2. *Let $\mathbf{M}$ be a left proper, combinatorial model category. A map $(u,f) : (S,\mathbf{X}) \to (T,\mathbf{Y})$ of $\mathbf{M}_{*}^{\Delta^{op}Set}$ is a*

(a) **fibred projective weak equivalence** *if u is bijective and $f : \mathbf{X} \to u^* \mathbf{Y}$ is a weak equivalence in $\mathbf{M}_{*,p,S}^{\Delta^{op}S}$;*

(b) **fibred projective cofibration** *if $u_! \mathbf{X} \to \mathbf{Y}$ is a cofibration in $\mathbf{M}_{*,p,S}^{\Delta^{op}T}$;*

(c) **fibred projective fibration** *if $f : \mathbf{X} \to u^* \mathbf{Y}$ is a fibration in $\mathbf{M}_{*,p,S}^{\Delta^{op}S}$.*

A map which is both a fibred projective weak equivalence and fibred projective cofibration is called isotrivial cofibration in [11, 14.3].

Theorem 6.3. *Let $\mathbf{M}$ be a left proper, combinatorial model category. The classes of fibred projective weak equivalences, fibred projective cofibrations and fibred projective fibrations form a model category structure on $\mathbf{M}_{*}^{\Delta^{op}Set}$. We denote this model structure by $f\mathbf{M}_{*,p,S}^{\Delta^{op}Set}$. $f\mathbf{M}_{*,p,S}^{\Delta^{op}Set}$ is a left Bousfield localization of $f\mathbf{M}_{*,p}^{\Delta^{op}Set}$.*

Proof. The proof is the same as for theorem 5.3, using the fact that if $f : S \to T$ is a function, the adjoint pair $f'_! : \mathbf{M}_{*,p,S}^{\Delta^{op}S} \rightleftarrows \mathbf{M}_{*,p,S}^{\Delta^{op}T} : f^*$ is a Quillen pair. $\qquad\square$

The next result was proved in the case when $\mathbf{M}$ is the category of simplicial sets by A. Joyal (unpublished). It is a straightforward application of theorem 3.1.

Theorem 6.4. *Let $\mathbf{M}$ be a left proper, combinatorial cartesian model category with cofibrant unit, having a set of generating cofibrations with cofibrant domains and satisfying the monoid axiom of [9]. Regard $\mathbf{M}$-$\mathbf{Cat}$ as having the fibred model structure [12, 4.2]. Then the adjoint pair*

$$ L : \mathbf{M}_{*,p,S}^{\Delta^{op}Set} \rightleftarrows \mathbf{M}\text{-}\mathbf{Cat} : N $$

is a Quillen equivalence.

7. Appendix: Two facts about left Bousfield localization

Let $\mathbf{M}$ be a model category and let C be a class of maps of $\mathbf{M}$. We denote by $L_C \mathbf{M}$ the left Bousfield localization of $\mathbf{M}$ with respect to C [6, 3.3.1(1)]. Recall that $\mathbf{S}$ denotes the category of simplicial sets.

Proposition 7.1. *A left proper, combinatorial monoidal model category with cofibrant unit and having a set of generating cofibrations with cofibrant domains is Quillen equivalent to a monoidal simplicial model category via a strong monoidal adjunction.*

Proof. Let $\mathbf{M}$ be as in the hypothesis, with unit I. We regard $\mathbf{M}^{\Delta^{op}}$ as having the Reedy model structure. By [1, Theorem 3.51] $\mathbf{M}^{\Delta^{op}}$ is a monoidal model category with cofibrant unit. Let $y : \Delta \to \mathbf{M}^{\Delta^{op}}$ be the functor $y([n]) = \underset{\Delta(-,[n])}{\sqcup} I$ and $\mathfrak{S}$ be the set of maps $\{y([n]) \to y([0])\}_{[n]\in\Delta}$ of $\mathbf{M}^{\Delta^{op}}$. The model structure $L_{\mathfrak{S}}\mathbf{M}^{\Delta^{op}}$ is a particular case of the one obtained by D. Dugger [5, Theorem 5.7], and it is also the left Bousfield localization of $\mathbf{M}^{\Delta^{op}}$ with respect to $\mathfrak{S}$ enriched over $\mathbf{M}$ [1, Definition 4.42 and Theorem 4.46]. By [1, Proposition 4.47] $L_{\mathfrak{S}}\mathbf{M}^{\Delta^{op}}$ is a monoidal model category. Among other things, Dugger proves that $L_{\mathfrak{S}}\mathbf{M}^{\Delta^{op}}$ is a simplicial model category. It follows that $L_{\mathfrak{S}}\mathbf{M}^{\Delta^{op}}$ is a monoidal simplicial model category (also known as monoidal $\mathbf{S}$-model category). By [5, Theorem 6.1]

$$cst : \mathbf{M} \rightleftarrows L_{\mathfrak{S}}\mathbf{M}^{\Delta^{op}} : ev_0$$

is a Quillen equivalence, where cst is the constant simplicial object functor and ev_0 is the evaluation at $[0]$. $\square$

Lemma 7.2. [5, Proof of Theorem 5.7(a)] *Let $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ be two model category structures on a category $\mathbf{M}$ such that the identity pair $Id : \mathbf{M}^{(1)} \rightleftarrows \mathbf{M}^{(2)} : Id$ is a Quillen pair and the weak equivalences of $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ are the same. Let C be a class of maps of $\mathbf{M}$. Then the weak equivalences of the left Bousfield localizations of $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ with respect to C are the same.*

Proof. Part of the hypothesis implies that every weak equivalence of $L_C\mathbf{M}^{(1)}$ is a weak equivalence of $L_C\mathbf{M}^{(2)}$. Conversely, let $f : X \to Y$ be a weak equivalence of $L_C\mathbf{M}^{(2)}$. Let $\bar{f} : \bar{X} \to \bar{Y}$ be a fibrant approximation to f in $L_C\mathbf{M}^{(1)}$ and $\tilde{f} : \tilde{X} \to \tilde{Y}$ a fibrant approximation to $\bar{f}$ in $\mathbf{M}^{(2)}$. It follows that $\tilde{X}$ and $\tilde{Y}$ are C-local with respect to $\mathbf{M}^{(2)}$, therefore $\tilde{f}$ is a weak equivalence of $\mathbf{M}^{(i)}, i \in \{1,2\}$. But then f is a weak equivalence of $L_C\mathbf{M}^{(1)}$. $\square$

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