Partial tense MV-algebras

Jan Paseka
Department of Mathematics and Statistics
Masaryk University
Brno, Czech Republic
paseka@math.muni.cz

Michal Botur
Department of Algebra and Geometry
Palacký University Olomouc
Olomouc, Czech Republic
michal.botur@upol.cz

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Outline

1. Introduction
2. Basic notions and definitions
3. Partial semi-states on MV-algebras
4. Partial functions between MV-algebras and their construction
5. The main theorem and its applications

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Introduction

For MV-algebras, the so-called tense operators were already introduced by Diaconescu and Georgescu. Tense operators express the quantifiers “it is always going to be the case that” and “it has always been the case that” and hence enable us to express the dimension of time in the logic.

A crucial problem concerning tense operators is their representation. Having a MV-algebra with tense operators, Diaconescu and Georgescu asked if there exists a frame such that each of these operators can be obtained by their construction for $[0, 1]$. We solved this problem for semisimple MV-algebras, i.e. those having a full set of MV-morphisms into a standard MV-algebra $[0, 1]$.

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Tense operators on $[0, 1]$

Tense operators were used to express the dimension of time in logics.

- Let T be a time scale,
- then elements $f(t)$ from $[0, 1]^T$ correspond to the evaluation of the validity of the formula f in time.

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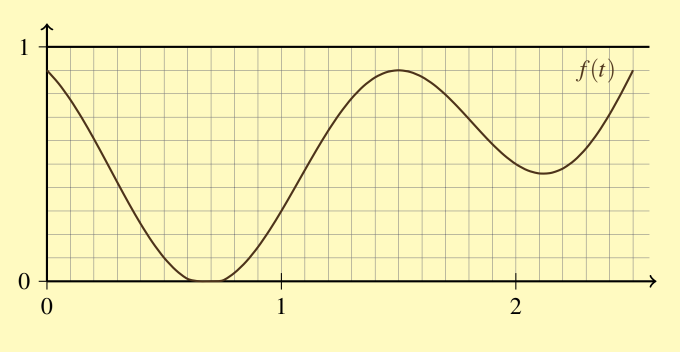

Tense operators on $[0,1]$

On the time scale T we will introduce a relation $R \subseteq T^2$.

- $x\,R\,y$ means that the moment x is before the moment y.

Moreover, we introduce operators G a H on $[0,1]^T$ as follows:

- Gf means that f will be true in future with at least the same degree as f is now.
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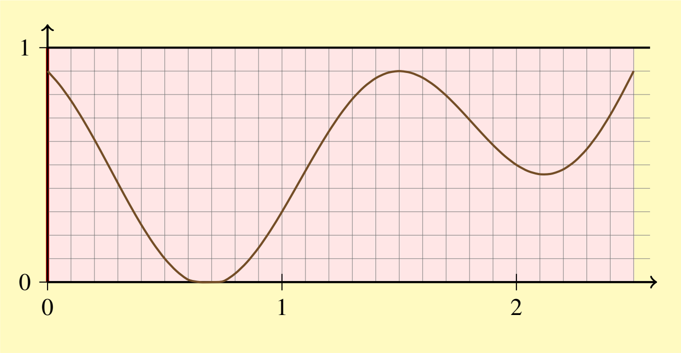

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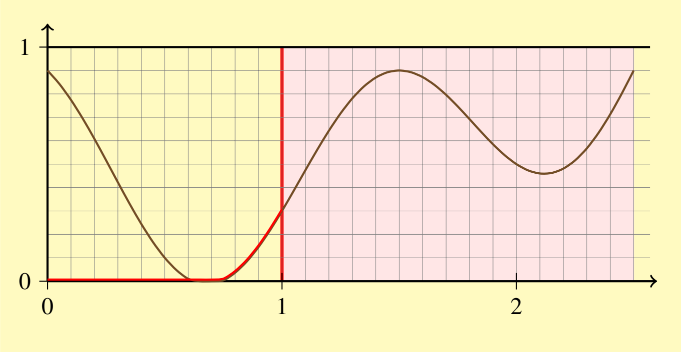

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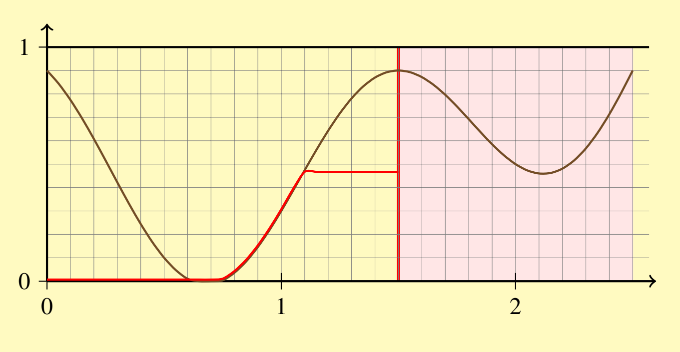

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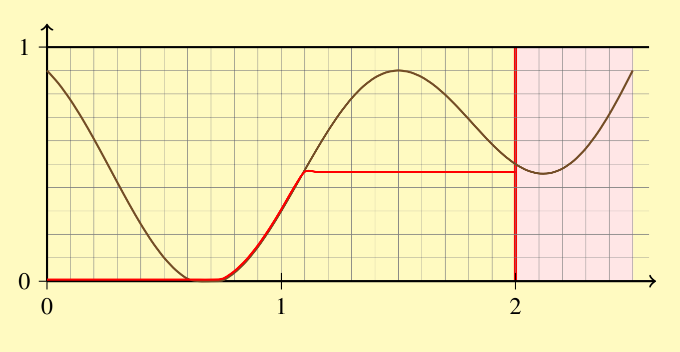

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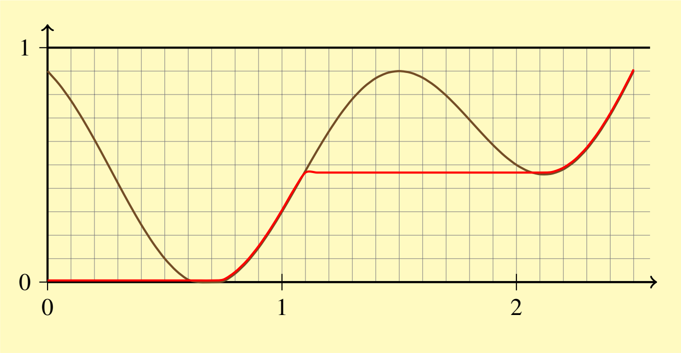

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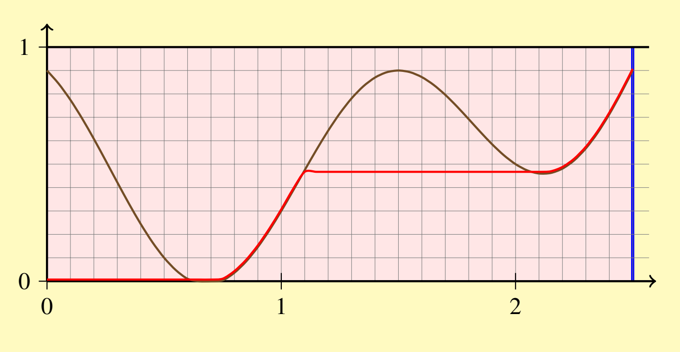

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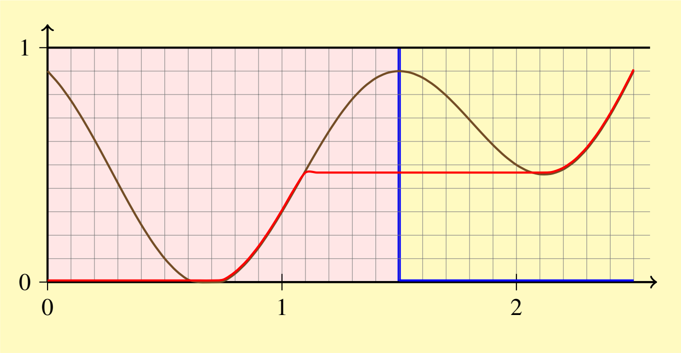

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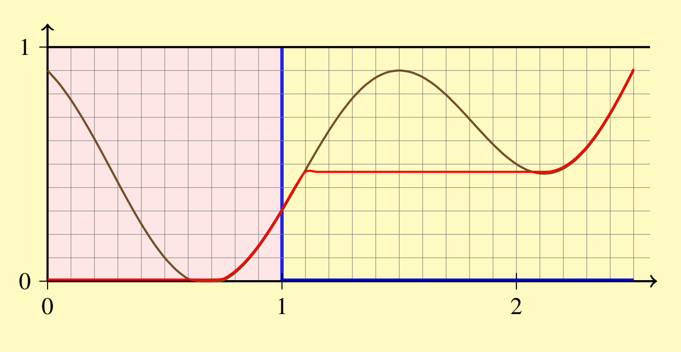

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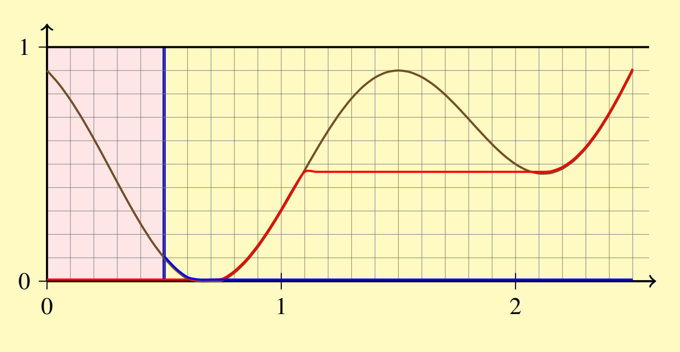

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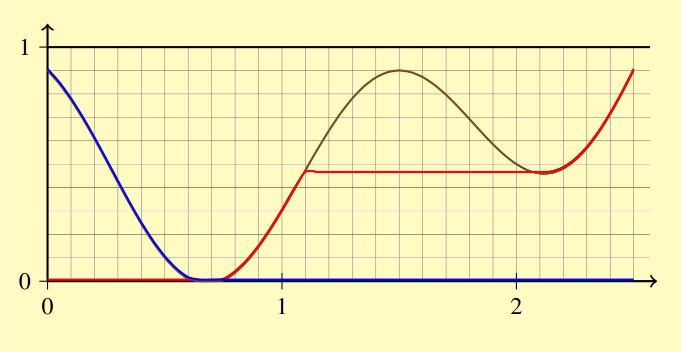

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Basic definition – MV-algebras

Definition (Chang, 1958)

An ***MV-algebra*** $\mathbf{M} = (M; \oplus, \odot, \neg, 0, 1)$ is a structure where $\oplus$ is associative and commutative with neutral element 0, and, in addition, $\neg 0 = 1, \neg 1 = 0, x \oplus 1 = 1, x \odot y = \neg(\neg x \oplus \neg y)$, and $y \oplus \neg(y \oplus \neg x) = x \oplus \neg(x \oplus \neg y)$ for all $x, y \in M$.

MV-algebras are a natural generalization of Boolean algebras. Namely, whilst Boolean algebras are algebraic semantics of Boolean two-valued logic, MV-algebras are algebraic semantics for Łukasiewicz many valued logic.

Example

An example of a MV-algebra is the real unit interval $[0, 1]$ equipped with the operations

$$\neg x = 1 - x, x \oplus y = \min(1, x + y), x \odot y = \max(0, x + y - 1)$$

We refer to it as a *standard MV-algebra*.

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On every MV-algebra $\mathbf{M}$, a partial order $\leq$ is defined by the rule

$$x \leq y \Longleftrightarrow \neg x \oplus y = 1.$$

In this partial order, every MV-algebra is a distributive lattice bounded by 0 and 1.

An MV-algebra is said to be *linearly ordered* (or a *MV-chain*) if the order is linear.

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Basic definitions – MV-morphisms and filters

Morphisms of MV-algebras (shortly *MV-morphisms*) are defined as usual, they are functions which preserve the binary operations $\oplus$ and $\odot$, the unary operation $\neg$ and nullary operations 0 and 1.

A *filter* of a MV-algebra $\mathbf{M}$ is a subset $F \subseteq M$ satisfying:
(F1) $1 \in F$
(F2) $x \in F,\ y \in M,\ x \leq y \ \Rightarrow\ y \in F$
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A filter is said to be *proper* if $0 \notin F$. Note that there is one-to-one correspondence between filters and congruences on MV-algebras.

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Basic definitions – Prime and maximal filters

A filter Q is *prime* if it satisfies the following conditions:

(P1) $0 \notin Q$.
(P2) For each x, y in M such that $x \vee y \in Q$, either $x \in Q$ or $y \in Q$.
In this case the corresponding factor MV-algebra $\mathbf{M}/Q$ is linear.

A filter U is *maximal* (and in this case it will be also called an ultrafilter) if $0 \notin U$ and for any other filter F of $\mathbf{M}$ such that $U \subseteq F$, then either $F = M$ or $F = U$. There is a one-to-one correspondence between ultrafilters and MV-morphisms from $\mathbf{M}$ into $[0, 1]$.

An MV-algebra $\mathbf{M}$ is called *semisimple* if the intersection of all its maximal filters is $\{1\}$.

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Basic definitions – states

We say that a *state* on a MV-algebra $\mathbf{M}$ is any mapping $s : M \to [0,1]$ such that (i) $s(1) = 1$, and (ii) $s(a \oplus b) = s(a) + s(b)$ whenever $a \odot b = 0$.

A state s is *extremal* if, for all states s_1, s_2 such that $s = \lambda s_1 + (1-\lambda)s_2$ for $\lambda \in (0,1)$ we conclude $s = s_1 = s_2$.

We recall that a state s is extremal iff $\{a \in M : s(a) = 1\}$ is an ultrafilter of $\mathbf{M}$ iff $s(a \oplus b) = \min\{s(a)+s(b), 1\}$, $a, b \in M$ iff s is a morphism of MV-algebras.

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Partial semi-states on MV-algebras

In this section we characterize the restriction of arbitrary meets of MV-morphism to a suitable domain into a unit interval as so-called partial semi-states.

Definition

Let $\mathbf{A}$ be an MV-algebra. A partial map $s : \mathbf{A} \to [0,1]$ is called *a partial semi-state on* $\mathbf{A}$ if

(i) $1 \in \mathrm{Dom}(s)$ and $s(1) = 1$,

(ii) $x, y \in \mathrm{Dom}(s)$ implies $x \wedge y \in \mathrm{Dom}(s)$ and $s(x \wedge y) = s(x) \wedge s(y)$,

(iii) $x \in \mathrm{Dom}(s)$ implies $x \odot x \in \mathrm{Dom}(s)$ and $s(x) \odot s(x) = s(x \odot x)$,

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Partial semi-states and MV-morphism

Note that any MV-morphism s from an MV-algebra $\mathbf{A}$ into a unit interval restricted to a set $D \subseteq A$ such that

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is a partial semi-state, i.e., $s_{/D}$ satisfies conditions (i)-(iv) from the preceding page.

Partial semi-states and MV-morphism

Note that any MV-morphism s from an MV-algebra $\mathbf{A}$ into a unit interval restricted to a set $D \subseteq A$ such that

(a) $1 \in D$,

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Meets of MV-morphism

Lemma

Let $\mathbf{A}$ be an MV-algebra, T a non-empty set of partial semi-states on $\mathbf{A}$ with the same domain D. Then the point-wise meet $t = \bigwedge T : D \to [0,1]$ is a partial semi-state on $\mathbf{A}$.

Lemma

Let $\mathbf{A}$ be an MV-algebra, s, t partial semi-states on $\mathbf{A}$ with the same domain D. Then $t \leq s$ iff $t(x) = 1$ implies $s(x) = 1$ for all $x \in D$.

Proposition

Let $\mathbf{A}$ be an MV-algebra, t a partial semi-state on $\mathbf{A}$ with a domain D and $T_t = \{s : \mathbf{A} \to [0,1] \mid s$ is an MV-morphism, $s_{/D} \geq t\}$. Then $t = (\bigwedge T_t)_{/D} = \bigwedge (T_t)_{/D}$.

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Any partial semi-state on the standard MV-algebra $[0,1]$ is the partial identity

Corollary

The only partial semi-state s on an MV-algebra $\mathbf{A}$ with a domain D such that $0 \in D$ and $s(0) \neq 0$ is the partial constant function $s(x) = 1$ for all $x \in D$.

Corollary

The only partial semi-state s on the standard MV-algebra $[0,1]$ with a domain D such that $s \neq \mathbf{1}_{/D}$ is the partial identity function id_D.

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2 Basic notions and definitions

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4 Partial functions between MV-algebras and their construction

5 The main theorem and its applications

Partial functions between MV-algebras

This section studies the notion of a partial fm-function between MV-algebras. The main purpose of this section is to establish in some sense a canonical construction of total fm-function between MV-algebras. This construction is an ultimate source of numerous examples.

Definition

By a **partial fm-function between MV-algebras** G is meant a partial function $G : \mathbf{A}_1 \to \mathbf{A}_2$ such that $\mathbf{A}_1 = (A_1; \oplus_1, \odot_1, \neg_1, 0_1, 1_1)$ and $\mathbf{A}_2 = (A_2; \oplus_2, \odot_2, \neg_2, 0_2, 1_2)$ are MV-algebras and

(FM1) $1_1 \in \mathrm{Dom}(G)$ and $G(1_1) = 1_2$,

(FM2) $x, y \in \mathrm{Dom}(G)$ implies $x \wedge y \in \mathrm{Dom}(G)$ and $G(x \wedge y) = G(x) \wedge G(y)$,

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The construction of total fm-functions between MV-algebras I

By a **frame** is meant a triple (S, T, R) where S, T are non-void sets and $R \subseteq S \times T$.

Having an MV-algebra $\mathbf{M} = (M; \oplus, \odot, \neg, 0, 1)$ and a non-void set T, we can produce the direct power $\mathbf{M}^T = (M^T; \oplus, \odot, \neg, o, j)$ where the operations $\oplus$, $\odot$ and $\neg$ are defined and evaluated on $p, q \in M^T$ componentwise. Moreover, o, j are such elements of M^T that $o(t) = 0$ and $j(t) = 1$ for all $t \in T$. The direct power $\mathbf{M}^T$ is again an MV-algebra.

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The construction of total fm-functions between MV-algebras II

Theorem

Let $\mathbf{M}$ be a linearly ordered complete MV-algebra, (S,T,R) be a frame and G^* be a map from M^T into M^S defined by

$$G^*(p)(s) = \bigwedge\{p(t) \mid t \in T, sRt\},$$

for all $p \in M^T$ and $s \in S$. Then G^* is a total fm-function between MV-algebras which has a left adjoint P^*.

In this case, for all $q \in M^S$ and $t \in T$,

$$P^*(q)(t) = \bigvee\{q(s) \mid s \in T, sRt\}.$$

We say that $G^* : \mathbf{M}^T \to \mathbf{M}^S$ is the *canonical total fm-function between MV-algebras induced by the frame (S,T,R) and the MV-algebra $\mathbf{M}$.*

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Semisimple MV-algebras

Recall that

1. *semisimple* MV-algebras are just subdirect products of the simple MV-algebras,
2. any simple MV-algebra is uniquelly embeddable into the standard MV-algebra on the interval $[0, 1]$ of reals,
3. an MV-algebra is semisimple if and only if the intersection of the set of its maximal (prime) filters is equal to the set $\{1\}$,
4. any complete MV-algebra is semisimple.

A semisimple MV-algebra $\mathbf{A}$ is embedded into $[0, 1]^T$ where T is the set of all ultrafilters of $\mathbf{A}$ (morphisms from $\mathbf{A}$ into the standard MV-algebra) and $\pi_F(x) = x(F) = x/F \in [0, 1]$ for any $x \in \mathbf{S} \subseteq [0, 1]^T$ and any $F \in T$; here $\pi_F : [0, 1]^T \to [0, 1]$ is the respective projection onto $[0, 1]$.

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Main theorem

Theorem

Let $G : \mathbf{A}_1 \to \mathbf{A}_2$ be a partial fm-function between semisimple MV-algebras, T (S) a set of all MV-morphism from $\mathbf{A}_1$ ($\mathbf{A}_2$) into the standard MV-algebra $[0,1]$. Further, let (S, T, ρ_G) be a frame such that the relation $\rho_G \subseteq S \times T$ is defined by

$$s\rho_G t \text{ if and only if } s(G(x)) \leq t(x) \text{ for any } x \in \mathrm{Dom}(G).$$

Then G is representable via the canonical total fm-function $G^* : [0,1]^T \to [0,1]^S$ between MV-algebras induced by the frame (S, T, ρ_G) and the standard MV-algebra $[0,1]$, i.e., the following diagram of partial fm-functions commutes:

$$\begin{tikzcd}
\mathbf{A}_1 \arrow[r, "G"] \arrow[d, "i^T_{\mathbf{A}_1}"'] & \mathbf{A}_2 \arrow[d, "i^S_{\mathbf{A}_2}"] \\
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Tense operators on MV-algebras

Definition (Botur and Paseka, Diaconescu and Georgescu)

Let $\mathbf{M}$ be an MV-algebra with total fm-functions G and H on $\mathbf{M}$. The structure $(\mathbf{M}; G, H)$ is called a *tense MV-algebra* if the following condition is fulfilled:

(GH) $\quad x \leq G(\neg H(\neg x)), \quad x \leq H(\neg G(\neg x))$, for all $x \in M$.

G	"It will always be the case that ..."
$P = \neg \circ H \circ \neg$	"It has at some time been the case that ..."
H	"It has always been the case that ..."
$F = \neg \circ G \circ \neg$	"It will at some time be the case that ..."

P and F are known as the *weak tense operators*, while H and G are known as the *strong tense operators*.

Moreover, P is a left adjoint to G and F is a left adjoint to H.

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Diaconescu and Georgescu formulated the following open problem:

Characterize those tense MV-algebras $(\mathbf{M};G,H)$ *such that* $i_{\mathbf{M}}:(\mathbf{M};G,H)\to([0,1]^T;G^*,H^*)$ *is a morphism of tense MV-algebra*, i.e., the following diagrams of fm-functions commute:

$$\begin{tikzcd} \mathbf{M} \arrow[r, "G"] \arrow[d, "i^T_{\mathbf{M}}"'] & \mathbf{M} \arrow[d, "i^T_{\mathbf{M}}"] \\ {[0,1]^T} \arrow[r, "G^*"'] & {[0,1]^T} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{M} \arrow[r, "H"] \arrow[d, "i^T_{\mathbf{M}}"'] & \mathbf{M} \arrow[d, "i^T_{\mathbf{M}}"] \\ {[0,1]^T} \arrow[r, "H^*"'] & {[0,1]^T} \end{tikzcd}\ .$$

Tense operators on MV-algebras

Theorem (Representation theorem for tense MV-algebras)

For any semisimple tense MV-algebra $(\mathbf{M}; G, H)$*,* $(\mathbf{M}; G, H)$ *is embeddable via the morphism* $i_{\mathbf{M}}$ *of tense MV-algebras into the canonical tense MV-algebra* $\mathbf{L}_{G,H} = ([0,1]^T; G^*, H^*)$ *with tense operators* G^*, H^* *induced by the canonical frames* (T, R_G)*,* (T, R_H) *and the standard MV-algebra* $[0,1]$*.*

Further $R_G = (R_H)^{-1}$ *and, for all* $x \in M$ *and for all* $s \in T$*,* $s(G(x)) = G^*((t(x))_{t\in T})(s)$ *and* $s(H(x)) = H^*((t(x))_{t\in T})(s)$*, i.e., the following diagrams of fm-functions commute:*

$$\begin{tikzcd}
\mathbf{M} \arrow[r, "G"] \arrow[d, "i^T_{\mathbf{M}}"'] & \mathbf{M} \arrow[d, "i^T_{\mathbf{M}}"] \\
{[0,1]^T} \arrow[r, "G^*"'] & {[0,1]^T}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{M} \arrow[r, "H"] \arrow[d, "i^T_{\mathbf{M}}"'] & \mathbf{M} \arrow[d, "i^T_{\mathbf{M}}"] \\
{[0,1]^T} \arrow[r, "H^*"'] & {[0,1]^T}
\end{tikzcd}\ .$$

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Thank you for your attention.