Weak wreath products and weak quantum duplicates

Esperanza López Centella

University of Novi Sad

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Classical

Factorization problem

A **factorization structure** of an algebra C by means of algebras A and B is a pair (i_A, i_B) of algebra morphisms $i_A : A \to C$, $i_B : B \to C$ such that $B \otimes A \cong C$ via $\mu_C(i_A \otimes i_B)$.

Factorization structures $\longleftrightarrow$ Distributive laws

one-to-one

A **distributive law** of A over B is a linear map $\tau : A \otimes B \to B \otimes A$ obeying

$$\begin{aligned} \tau(\mu_A \otimes B) &= (B \otimes \mu_A)(\tau \otimes A)(A \otimes \tau) \\ \tau(A \otimes \mu_B) &= (\mu_B \otimes A)(B \otimes \tau)(\tau \otimes B) \\ \tau(a \otimes 1) &= 1 \otimes a, \quad \tau(1 \otimes b) = b \otimes 1 \end{aligned}$$

for any $a \in A, b \in B$.

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for any $a \in A, b \in B$.

Factorization problem

For any distributive law $\tau : A \otimes B \to B \otimes A$, the map

$$\mu_\tau := (\mu_B \otimes \mu_A)(B \otimes \tau \otimes A)$$

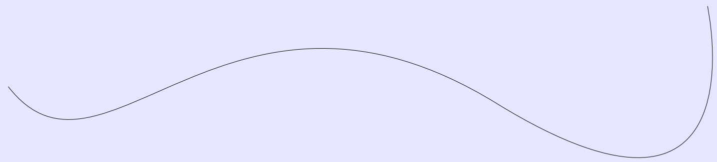
defines an associative and unital multiplication on $B \otimes A$.

The algebra $(B \otimes A, \mu_\tau)$ is called **wreath product** of A and B and denoted by $B \otimes_\tau A$.

Proposition. For any factorization structure of an algebra C by means of algebras A and B, there exists an unique distributive law $\tau : A \otimes B \to B \otimes A$ such that C is isomorphic to $B \otimes_\tau A$ as a wreath product.

Quantum duplicates

Consider a manifold M representing some physical system:



$$A = \mathcal{C}^\infty(M)$$

From a dual point of view, this manifold can be also represented by some algebra of functions, for instance, $A = \mathcal{C}^\infty(M)$ (the algebra of smooth functions on M) over some base field k.

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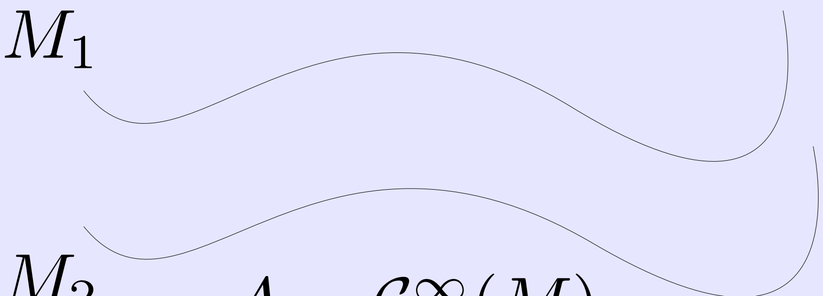

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Commutative duplicate

$$A \times A \cong A \otimes k^2$$

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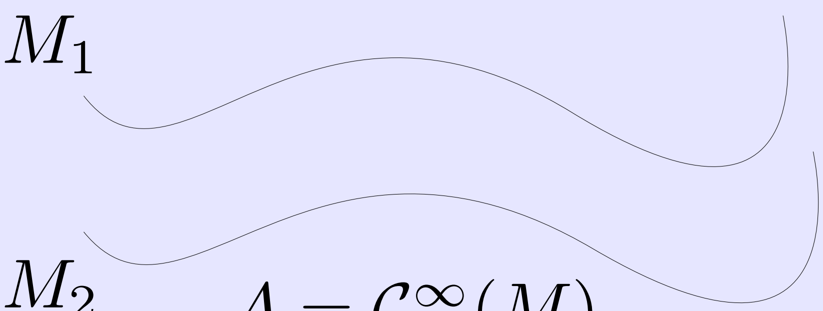

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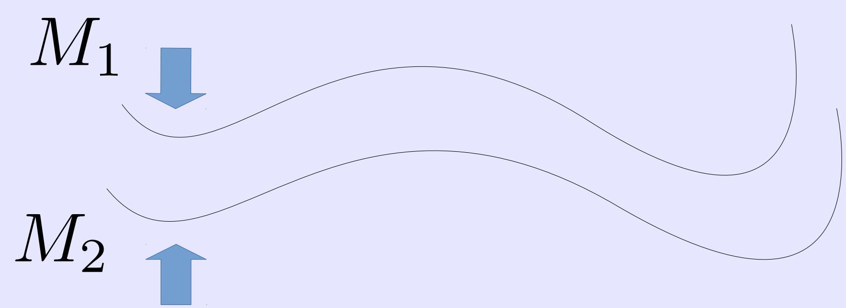
Closer and closer!



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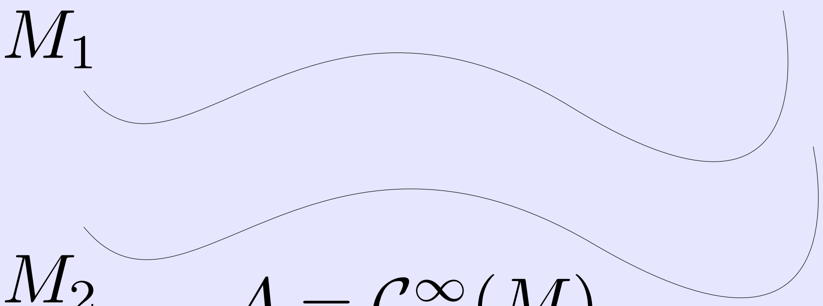

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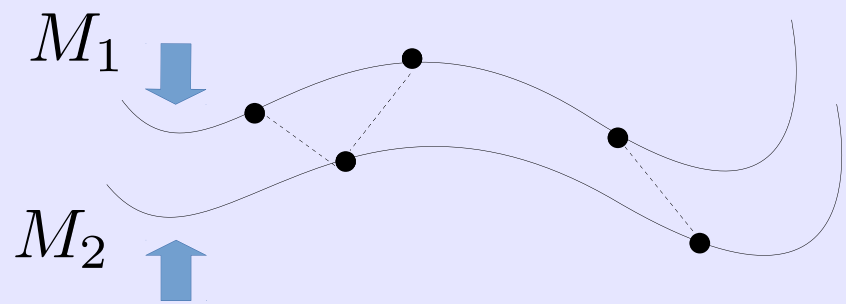

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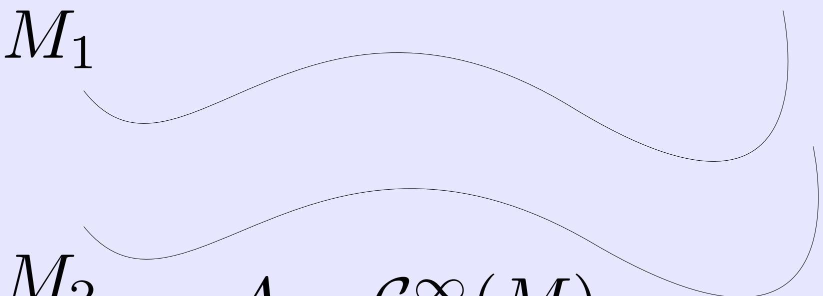

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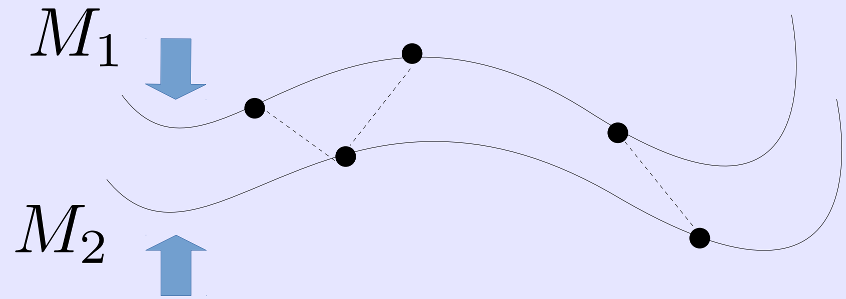

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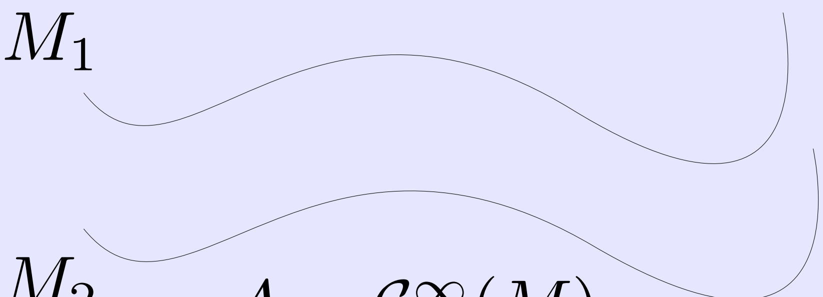

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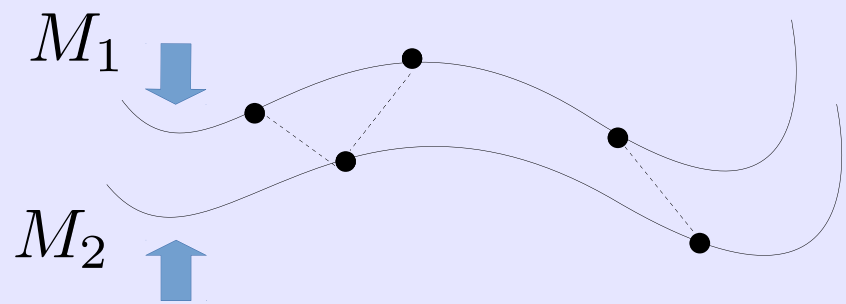

Commutative duplicate

$A \times A \cong A \otimes k^2$

Non-commutative duplicate

$A \otimes_\tau k^2$

Deformation of $A \otimes k^2$ with similar structural properties

Quantum duplicates

A **quantum duplicate** of an algebra A is a wreath product $B \otimes_\tau A$, where $dim_k(B) = 2$.

$$dim_k(B) = 2 \Longrightarrow B = k[x]/(p(x)), \;\; p(x) = x^2 - cx + d, \;\; c, d \in k.$$

Proposition (*Cortadellas et al.*). The set of distributive laws $\tau : k[\xi] \otimes B \to B \otimes k[\xi]$ is in one-to-one correspondence with the set of pairs (f, δ) obeying

- $f : B \to B$ is an endomorphism of algebras,
- $\delta : B \to B$ is a left f–derivation,
- $\delta^2 - c\delta = d(f^2 - 1)$,
- $f\delta + \delta f = c(f - f^2)$.

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Weak

Weak factorization problem

A **weak factorization structure** of an algebra C by means of algebras A and B is a triple (i_A, i_B, i) of algebra morphisms $i_A : A \to C, i_B : B \to C$ such that $i : C \to B \otimes A$ is (B, A)–bimodule section of $\mu_C(i_A \otimes i_B)$.

Weak distributive laws ⟷ Weak factorization structures

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Weak factorization problem

For any weak distributive law $\tau : A \otimes B \to B \otimes A$, the map

$$\mu_\Psi := (\mu_B \otimes \mu_A)(B \otimes \Psi \otimes A)$$

defines an associative multiplication on $B \otimes A$

The natural candidate, the element $1 \otimes 1$, fails to be the unit for this multiplication: For any $a \in A, b \in B$,

$$(a \otimes b)(1 \otimes 1)$$

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$$(a \otimes b)(1 \otimes 1) \quad = \quad a\underbrace{\Psi(b \otimes 1)}_{\neq \; 1 \otimes b} \quad \neq a \otimes b$$

Lemma. The element $\mathbf{1} = (1 \otimes 1)^2$ is a central idempotent for the multiplication μ_Ψ.

Weak factorization problem

Proposition *(Street)*. Let $\Psi : A \otimes B \to B \otimes A$ a weak distributive law. Then the linear map

$$\kappa_\Psi : B \otimes A \to B \otimes A, \quad \kappa_\Psi(b \otimes a) := (b \otimes 1)(1 \otimes a)$$

is an idempotent endomorphism of (non-unital) algebras and of (B, A)–bimodules and $\kappa_\Psi \Psi = \Psi \quad$ and $\quad \kappa_\Psi \mu_\Psi = \mu_\Psi$.

Denote by $B \otimes_\Psi A$ the range of κ_Ψ in $B \otimes A$.

➡ $(B \otimes_\Psi A, \mu_\Psi)$ is a unital algebra, termed as **weak wreath product** of A and B.

Weak factorization problem

Proposition *(Böhm & Gómez-Torrecillas).* For any weak distributive law $\Psi : A \otimes B \to B \otimes A$, the maps

$$i_B : A \to B \otimes_\Psi A, \quad b \otimes a \mapsto (b \otimes 1)(1 \otimes 1)$$
$$i_A : A \to B \otimes_\Psi A, \quad b \otimes a \mapsto (1 \otimes 1)(1 \otimes a)$$

and the inclusion $i : B \otimes_\Psi A \to B \otimes A$ constitutes a weak factorization of $B \otimes_\Psi A$.

Conversely, if (i_B, i_A, i) is a weak factorization of an algebra C by means of algebras A and B, then the map $\Psi := i\mu_C(i_A \otimes i_B)$ is a weak distributive law of A over B such that the corresponding κ_Ψ splits and $C \cong B \otimes_\Psi A$ as unital algebras.

Weak quantum duplicates

We call **weak quantum duplicate** a weak wreath product involving an arbitrary algebra B and a 2-dimensional algebra A.

The number of ways in which this 2-dimensional factor can be chosen depends on the field k.

More concretely, if k admits a degree 2 field extension k, then there are three non-isomorphic algebras of dimension 2 (over k):

- The trivial direct product k^2
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- The trivial direct product k^2 $\cong k[x]/ < x^2 - x >$
- The ring of dual numbers $\cong k[x]/ < x^2 >$
- Quadratic field extensions of k $\cong k[x]/ < p(x) >$
  $p(x)$ irreducible

Weak quantum duplicates

Aim: To give a characterization of

weak quantum duplicates

$\Updownarrow$

weak distributive laws

$\Psi : k[\xi] \otimes B \to B \otimes k[\xi]$, with $\xi^2 = c\xi - d$.

Characterization

Theorem. Let B be an algebra over k. The set of weak distributive laws $\Psi : k[\xi] \otimes B \to B \otimes k[\xi]$ is in one-to-one correspondence with the set of quadruples (f, δ, p, q), where $f, \delta : B \to B$ are linear maps and p, q are elements of B obeying:

- $B \to \mathcal{M}_2(B), \quad b \mapsto \begin{pmatrix} bp & bq \\ \delta(b) & f(b) \end{pmatrix}$ is multiplicative.
- $\begin{cases} \diamond\, p = p^2 - dq^2, \quad q = cq^2 + pq + qp \\ \diamond\, \delta^2 - c\delta = d(f^2 - \overline{p}) \\ \diamond\, f\delta + \delta f = c(f - f^2) - d\overline{q} \\ \diamond\, \delta = \delta\overline{p} - df\overline{q} = \overline{p}\delta - d\overline{q}f \\ \diamond\, f = cf\overline{q} + f\overline{p} + \delta\overline{q} = c\overline{q}f + \overline{q}\delta + \overline{p}f \\ \diamond\, f(1) = p + cq, \quad \delta(1) = -dq \end{cases}$

denoting $\overline{p}, \overline{q}$ the maps $B \to B$ sending any $b \in B$ to bp and bq.

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(p,q)-equations

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Characterization

Remarks.

- The first 'multiplicativity' condition is independent of the 2–dimensional factor.

- This characterization generalizes that one by *Cibils* and *Cortadellas et al.*. Indeed, (non-weak) quantum duplicates are labeled by those quadruples of the type $(1, 0, f, \delta)$.

- $dim(B \otimes_\Psi k[\xi]) \neq 2 \cdot dim(B)$ (in general).

Application

Aim: Classify (up to isomorphism) the weak wreath products of two 2–dimensional algebras.

Method: Determine the weak distributive laws
$\Psi : k[\xi] \otimes k[\eta] \to k[\eta] \otimes k[\xi]$, with $\xi^2 = c\xi - d,\ \eta^2 = C\eta - D$.

Cases:

k does NOT admit quadratic extensions

	(c, d, C, D)
$k^2 \otimes_\Psi k^2$	$(1, 0, 1, 0)$
$k[\zeta] \otimes_\Psi k[\zeta]$	$(0, 0, 0, 0)$
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Application

Aim: Classify the weak wreath products over two 2–dimensional algebras.

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k admits quadratic extensions

plus!

$l \otimes_\Psi k^2$

$l \otimes_\Psi k[\zeta]$

$l \otimes_\Psi l'$

$k^2 \otimes_\Psi l$

$k[\zeta] \otimes_\Psi l$

Brief comments

- For most of the cases, we use mathematical software to solve all the equations imposed on such weak distributive laws.

- (p,q)–equations are crucial within the system of equations. They are solved first. For each solution pair (p, q), the remaining equations are then solved, obtaining the admissible (f, δ).

- Charasteristic of the field k matters (we distinguish $char(k) = 2$ or $char(k) \neq 2$).

2 x 2 = 3 and

commutative x commutative = non-commutative

Set $u_1 = (1,0)$ and $u_2 = (0,1)$ and consider the following bases of k^2 and the ring $\mathcal{T}$ of upper triangular matrices:

$$B_{k^2 \otimes k^2} = \{u_1 \otimes u_1,\ u_1 \otimes u_2,\ u_2 \otimes u_1,\ u_2 \otimes u_2\}$$

$$\mathrm{B}_{\mathcal{T}} = \{x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \ y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ \ \alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\}.$$

We have the following two homomorphisms of unital k–algebras:

$$\begin{aligned} k^2 &\xrightarrow{\ \beta\ } \mathcal{T} \\ u_1 &\longmapsto \alpha + x \\ u_2 &\longmapsto y - \alpha \end{aligned} \qquad\qquad \begin{aligned} k^2 &\xrightarrow{\ \gamma\ } \mathcal{T} \\ u_1 &\longmapsto y \\ u_2 &\longmapsto x \end{aligned}$$

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Then,

$$\nu : k^2 \otimes k^2 \xrightarrow{\beta\otimes\gamma} \mathcal{T} \otimes \mathcal{T} \xrightarrow{\mu_{\mathcal{T}}} \mathcal{T}$$

and

$$\iota : \mathcal{T} \longrightarrow K^2 \otimes k^2$$

$$x \longmapsto u_1 \otimes u_2, \quad y \longmapsto u_1 \otimes u_1 + u_2 \otimes u_1, \quad \alpha \longmapsto u_1 \otimes u_1$$

hold $\nu \circ \iota = id_{\mathcal{T}}$, and ι is an homomorphism of k^2–bimodules. That is, ν is an splitting epimorphism of k^2–bimodules.

$\overset{\text{Theorem}}{\Longrightarrow}$ $\mathcal{T} \cong k^2 \otimes_\Psi k^2$, for the following weak distributive law:

$$\Psi : k^2 \otimes k^2 \xrightarrow{\gamma\otimes\beta} \mathcal{T} \otimes \mathcal{T} \xrightarrow{\mu_{\mathcal{T}}} \mathcal{T} \xrightarrow{\iota} k^2 \otimes k^2.$$


Děkuji moc ☺!
Thank you
very much!