Complete Ω-lattices

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G.P. Monro, *Quasitopoi, logic and Heyting-valued models*, Journal of pure and applied algebra, (1986) 42(2), 141-164,
E. Palmgren, S.J. Vickers, *Partial Horn logic and cartesian categories*, Annals of Pure and Applied Logic, (2007) 145(3), 314-353.

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U. Höhle, *Fuzzy sets and sheaves. Part I: basic concepts*, Fuzzy Sets and Systems, (2007) 158(11),
S. Gottwald, *Universes of fuzzy sets and axiomatizations of fuzzy set theory, Part II: Category theoretic approaches*, Studia Logica, (2006) 84(1), 23-50. 1143-1174,
R. Bělohlávek, *Fuzzy equational logic*, Archive for Mathematical Logic 41.1 (2002): 83-90,
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This is not a classical lattice, still classical lattices appear as special quotients with respect to the Ω-valued equality.

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A closure system is a complete lattice under inclusion.

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Under the above conditions, we say that (M, E, R) is an **Ω-poset**.

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Theorem

Let (M, E, R) be an Ω-poset. Then for every $p \in \Omega$, the quotient structure $(\mu_p/E_p, \leq)$ is a poset, where the relation $\leq$ is defined by

$$[x]_p \leq [y]_p \text{ if and only if } (x, y) \in R_p.$$

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Then an element $u \in M$ is a **pseudo-supremum** of A, if for every $p \in \Omega$, $p \leq \bigwedge(\mu(x) \mid x \in A)$, the following hold:

(*i*) u is an upper bound of A and

(*ii*) if there is $u_1 \in M$ such that $p \leq R(a, u_1)$ for every $a \in A$, then $p \leq R(u, u_1)$.

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Dually, an element $v \in M$ is a **pseudo-infimum** of A, if for every $p \in \Omega$, $p \leq \bigwedge(\mu(x) \mid x \in A)$, the following hold:

(j) v is a lower bound of A and

(jj) if there is $v_1 \in M$ such that $p \leq R(v_1, a)$ for every $a \in A$, then $p \leq R(v_1, v)$.

Proposition

Let (M, E, R) be an Ω-poset, let $A \subseteq M$ and $u \in M$ a pseudo-supremum (pseudo-infimum) of $A \subseteq M$. Then $v \in M$ is also a pseudo-supremum (pseudo-infimum) of $A \subseteq M$, if and only if $\bigwedge(\mu(x) \mid x \in A) \leq E(u, v)$.

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For an arbitrary subset $A \subseteq M$, if a pseudo-supremum (pseudo-infimum) exists it is generally not unique.

Two pseudo-suprema u, v of A belong to the same equivalence class μ_p/E_p for every $p \leq \bigwedge(\mu(x) \mid x \in A)$.

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In particular, if $A = M$, then the above elements t and b are said to be a **pseudo-top** and a **pseudo-bottom**, respectively, of the whole Ω-poset (M, E, R).

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A complete Ω-lattice possesses a pseudo-top and a pseudo bottom element.

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Let (M, E, R) be a complete Ω-lattice. Then, for every $p \in \Omega$, the poset $(\mu_p/E_p, \leq_p)$ is a complete lattice. In addition, for $A \subseteq M$, if c is a pseudo-infimum of A in (M, E, R), then $[c]_p$ is the infimum of $\{[a]_p \mid a \in A\}$ in the lattice $(\mu_p/E_p, \leq_p)$, for every $p \in \Omega$, such that $A \subseteq \mu_p$.
Analogously, if d is a pseudo-supremum of A, then $[d]_p$ is the supremum of $\{[a]_p \mid a \in A\}$ in $(\mu_p/E_p, \leq_p)$.

Theorem

Let (M, E, R) be an Ω-poset. Then it is a complete Ω-lattice if and only if for every $q \in \Omega$, the poset $(\mu_q/E_q, \leq_q)$ is a complete lattice, and the following holds: for all $A \subseteq M$, $p = \bigwedge(\mu(a) \mid a \in A)$, and $q \leq p$, we have

$$\inf\{[a]_{E_p} \mid a \in A\} \subseteq \inf\{[a]_{E_q} \mid a \in A\},$$

$$\textit{and} \quad \sup\{[a]_{E_p} \mid a \in A\} \subseteq \sup\{[a]_{E_q} \mid a \in A\},$$

where the infima (suprema) belong to the corresponding posets $(\mu_q/E_q, \leq_q)$ and $(\mu_p/E_p, \leq_p)$.

Theorem

An Ω*-poset* (M, E, R) *is a complete* Ω*-lattice, if the following conditions are fulfilled:*

(i) *a pseudo-infimum exists for every* $A \subseteq M$*;*

(ii) *every cut* μ_p*,* $p \in \Omega$*, possesses a pseudo-top element;*

(iii) *for all* $A \subseteq M$*,* $p = \bigwedge(\mu(a) \mid a \in A)$*, and* $q \leq p$*, if* $\sup\{[a]_{E_p} \mid a \in A\}$ *and* $\sup\{[a]_{E_q} \mid a \in A\}$ *exist in the posets* $(\mu_q/E_q, \leq_q)$ *and* $(\mu_p/E_p, \leq_p)$ *respectively, then*

$$\sup\{[a]_{E_p} \mid a \in A\} \subseteq \sup\{[a]_{E_q} \mid a \in A\}.$$

In the following we denote:

$$\Delta(f) := \{x \in M \mid (x, x) \in f\}.$$

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Theorem

Let $M \neq \emptyset$, and let $\mathcal{F} \subseteq \mathcal{P}(M^2)$ be a closure system over M^2 such that each $f \in \mathcal{F}$ is transitive and strict. Then the following hold.
(a) There is a complete lattice Ω and a mapping $R : M^2 \longrightarrow \Omega$ such that $\mathcal{F}$ is a collection of cuts of R and (M, E, R) is an Ω-poset, where $E : M^2 \longrightarrow \Omega$ is defined by
$E(x,y) = R(x,y) \wedge R(y,x)$.
(b) Let, in addition, for every $f \in \mathcal{F}$ and for every $A \subseteq \Delta_f$ there is an infimum and a supremum in the relational structure $(\Delta(f), f)$, and for $g \in \mathcal{F}$, such that $f \subseteq g$, the following hold:

if c is an infimum of A in $\Delta(f)$, then c is an infimum of A in $\Delta(g)$;

if c is a supremum of A in $\Delta(f)$, then c is a supremum of A in $\Delta(g)$.

Then, (M, E, R) is a complete Ω-lattice.

Ω-algebras

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An **Ω-algebra** is a pair $(\mathcal{A}, E)$, where $\mathcal{A} = (A, F)$ is an algebra with the set F of fundamental operations, and (A, E) is an Ω-set, where Ω-valued equality $E : M^2 \to \Omega$ fulfills

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$$\bigwedge_{i=1}^{n} E(x_i, y_i) \leqslant E(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)),$$ for an n-ary operation $f \in F$ – compatibility.

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$E(x, y) = E(x, x) = E(y, y)$ implies $x = y$ – strong separation.

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The following is straightforward.

For all $x_1, \ldots, x_n \in A$ and for an n-ary $f \in F$, μ fulfills
$$\bigwedge_{i=1}^{n} \mu(x_i) \leqslant \mu(f(x_1, \ldots, x_n))$$ – *compatibility.*

Identities

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If $(\mathcal{A}, E)$ is an Ω-algebra, and $u \approx v$ is an identity in the language of the algebra $\mathcal{A}$, then we say that $(\mathcal{A}, E)$ **fulfills** the identity $u \approx v$ if

$$\bigwedge_{i=1}^{n} \mu(x_i) \leqslant E(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)),$$

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If $(\mathcal{A}, E)$ is an Ω-algebra, and the algebra $\mathcal{A}$ satisfies an identity $u \approx v$, then also $(\mathcal{A}, E)$ satisfies this identity.

Cut properties

Cut properties

Theorem

Let $(\mathcal{A},\ E)$ be an Ω-algebra, and Σ a set of identities. Then, $(\mathcal{A},\ E)$ fulfils Σ if and only if for every $p \in \Omega$, the cut μ_p is a subalgebra of $\mathcal{A}$, the cut relation E_p is a congruence on μ_p, and the quotient structure μ_p/E_p satisfies Σ.

Ω-lattice as an algebra

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As already defined, $E : M^2 \to \Omega$ is an Ω-valued equality, and the function $\mu : M \to \Omega$ is given by $\mu(x) = E(x, x)$.

Then, $(\mathcal{M}, E)$ is an **Ω-lattice** if the following formulas hold:

$$\mu(x) \wedge \mu(y) \leqslant E(x \sqcap y, y \sqcap x)$$
$$\mu(x) \wedge \mu(y) \leqslant E(x \sqcup y, y \sqcup x)$$
$$\mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E((x \sqcap y) \sqcap z, x \sqcap (y \sqcap z))$$
$$\mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E((x \sqcup y) \sqcup z, x \sqcup (y \sqcup z))$$
$$\mu(x) \wedge \mu(y) \leqslant E((x \sqcap y) \sqcup x, x)$$
$$\mu(x) \wedge \mu(y) \leqslant E((x \sqcup y) \sqcap x, x).$$

Proposition

In an Ω-lattice $(\mathcal{M}, E)$ the idempotent laws $\mu(x) \leqslant E(x \sqcap x, x)$ and $\mu(x) \leqslant E(x \sqcup x, x)$ are fulfilled.

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Proposition

If $(\mathcal{M}, E)$ is an Ω-lattice, then the bigroupoid $\mathcal{M}$ is idempotent with respect to both operations.

If $\mu : M \to \Omega$, and $p \in \Omega$, then a ***p*-cut**, or a **cut** of μ is a subset μ_p of M defined by

$$\mu_p := \mu^{-1}(\uparrow p).$$

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Theorem

Let $\mathcal{M} = (M, \wedge, \vee)$ be a bigroupoid and E an Ω-equality on $\mathcal{M}$. Then, $(\mathcal{M}, E)$ is an Ω-lattice if and only if for every $p \in \Omega$, the cut μ_p is a subalgebra (sub-bigroupoid) of $\mathcal{M}$, the cut relation E_p is a congruence on μ_p and a quotient structure μ_p/E_p is a lattice.

Order on Ω-lattices

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Theorem

If $(\mathcal{M}, E)$ is an Ω-lattice, then the Ω-valued relation $R : M^2 \to L$, such that
$R(x, y) := \mu(x) \wedge \mu(y) \wedge E(x \sqcap y, x)$
is an Ω-valued order on $\mathcal{M}$. Moreover, for every $p \in \Omega$, the order on the lattice μ_p/E_p is induced by the cut R_p:
$[x]_{E_p} \leqslant [y]_{E_p}$ *if and only if* $(x, y) \in R_p$.

Let (M, E, R) be an Ω-poset, and $a, b \in M$. An element $c \in M$ is a **pseudo-infimum** of a, b, if for every $p \in \Omega$ such that $\mu(a) \wedge \mu(b) \geqslant p$, the following holds:

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(i) $\mu(c) \wedge R(c, a) \wedge R(c, b) \geqslant p$ and

for every $x \in M$

$\mu(x) \wedge R(x, a) \wedge R(x, b) \geqslant p$ implies $R(x, c) \geqslant p$.

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An element $d \in M$ is a **pseudo-supremum** of $a, b \in M$, if for every $p \in \Omega$ such that $\mu(a) \wedge \mu(b) \geqslant p$, the following holds:
(ii) $\mu(d) \wedge R(a, d) \wedge R(b, d) \geqslant p$ and
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$\mu(x) \wedge R(a, x) \wedge R(b, x) \geqslant p$ implies $R(d, x) \geqslant p$.

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for every $x \in M$
$\mu(x) \wedge R(a, x) \wedge R(b, x) \geqslant p$ implies $R(d, x) \geqslant p$.

Observe that for given $a, b \in M$, a pseudo-infimum and a pseudo-supremum, if they exist, are not unique in general.

We say that an Ω-poset (M, E, R) is an **Ω-lattice as an ordered structure**, if for every $a, b \in M$ there exist a pseudo-infimum and a pseudo-supremum.

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Proposition

Let (M, E, R) be an Ω-lattice and c, c_1 pseudo-infima of $a, b \in M$. If for $p \in \Omega$, $\mu(a) \wedge \mu(b) \geqslant p$, then $E(c, c_1) \geqslant p$. Analogously, if d, d_1 are pseudo-suprema of a, b and $\mu(a) \wedge \mu(b) \geqslant p$, then $E(d, d_1) \geqslant p$.

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Obviously, by this Proposition, pseudo-infima (suprema) of two element a, b from μ_p, belong to the same equivalence class in μ_p / E_p.

Theorem

Let (M, E, R) *be an* Ω*-lattice as an ordered structure. Then for every* $p \in \Omega$*, the poset* $(\mu_p/E_p, \leq_p)$ *is a lattice, where the relation* $\leq_p$ *on the quotient set* μ_p/E_p *is defined above.*

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Theorem

Let $\mathcal{M} = (M, \sqcap, \sqcup)$ *be a bi-groupoid,* $(\mathcal{M}, E)$ *an* Ω*-lattice as an algebra in which* E *is strongly separated, and* $R : M^2 \to \Omega$ *an* Ω*-valued relation on* M *defined by* $R(x, y) := E(x \sqcap y, x)$*. Then,* (M, E, R) *is an* Ω*-lattice as an ordered structure.*

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We define two binary operations, $\sqcap$ and $\sqcup$ on M as follows: for every pair a, b of elements from M, $a \sqcap b$ is an arbitrary, fixed pseudo-infimum of a and b, and $a \sqcup b$ is an arbitrary, fixed pseudo-supremum of a and b.

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Assuming Axiom of Choice, by which an element is chosen among all pseudo-infima (suprema) of a and b, the operations $\sqcap$ and $\sqcup$ on M are well defined.

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Assuming Axiom of Choice, by which an element is chosen among all pseudo-infima (suprema) of a and b, the operations $\sqcap$ and $\sqcup$ on M are well defined.

Theorem

If (M, E, R) is an Ω-lattice as an ordered structure, and $\mathcal{M} = (M, \sqcap, \sqcup)$ the bi-groupoid in which operations $\sqcap$, $\sqcup$ are introduced above, then $(\mathcal{M}, E)$ is an Ω-lattice as an algebra.

Example

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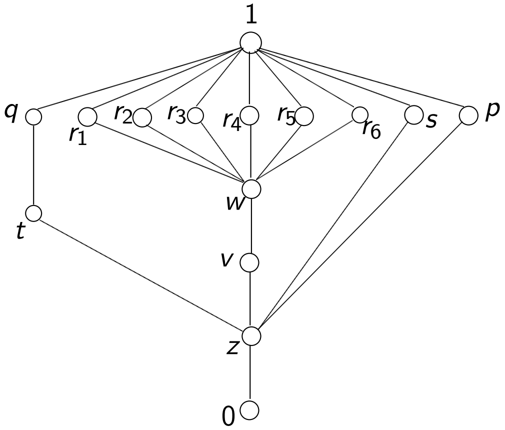

Lattice Ω

$$M = \{x_0, x_1, \dots, x_9, x_{10}\}$$

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R	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_0	p	z	z	z	z	z	z	z	z	z	p
x_1	0	q	q	0	0	0	0	0	0	z	0
x_2	0	t	q	0	0	0	0	0	0	z	0
x_3	0	0	0	r_1	w	w	w	w	w	z	0
x_4	0	0	0	w	r_2	w	w	z	z	z	0
x_5	0	0	0	w	w	r_3	w	w	w	z	0
x_6	0	0	0	w	w	w	r_4	w	w	z	0
x_7	0	0	0	z	z	z	z	r_5	w	z	0
x_8	0	0	0	z	z	z	z	v	r_6	z	0
x_9	0	0	0	0	0	0	0	0	0	s	0
x_{10}	z	z	z	z	z	z	z	z	z	z	p

$$E(x, y) = R(x, y) \wedge R(y, x)$$

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E	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_0	p	0	0	0	0	0	0	0	0	0	z
x_1	0	q	t	0	0	0	0	0	0	0	0
x_2	0	t	q	0	0	0	0	0	0	0	0
x_3	0	0	0	r_1	w	w	w	z	z	0	0
x_4	0	0	0	w	r_2	w	w	z	z	0	0
x_5	0	0	0	w	w	r_3	w	z	z	0	0
x_6	0	0	0	w	w	w	r_4	z	z	0	0
x_7	0	0	0	z	z	z	z	r_5	v	0	0
x_8	0	0	0	z	z	z	z	v	r_6	0	0
x_9	0	0	0	0	0	0	0	0	0	s	0
x_{10}	z	0	0	0	0	0	0	0	0	0	p

Quotient lattices:

Quotient lattices:
$\mu_z/E_z = \{\{x_0, x_{10}\}, \{x_1, x_2\}, \{x_3, \ldots, x_8\}, \{x_9\}\}$, – Boolean lattice;
$\mu_q/E_q = \{\{x_1\}, \{x_2\}\}$;
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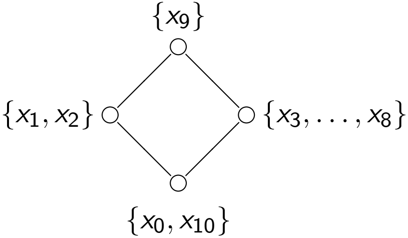

μ_z/E_z

μ_q/E_q

$\sqcap$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_0	x_0	x_0	x_0	x_0	x_0	x_0	x_0	x_0	x_0	x_0	x_0
x_1	x_0	x_1	x_1	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_1	x_{10}
x_2	x_0	x_1	x_2	x_0	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_2	x_{10}
x_3	x_0	x_{10}	x_{10}	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_{10}
x_4	x_0	x_{10}	x_{10}	x_3	x_4	x_3	x_4	x_4	x_4	x_4	x_{10}
x_5	x_0	x_{10}	x_{10}	x_3	x_3	x_5	x_5	x_5	x_5	x_5	x_{10}
x_6	x_0	x_{10}	x_{10}	x_3	x_4	x_5	x_6	x_4	x_6	x_6	x_{10}
x_7	x_0	x_{10}	x_{10}	x_3	x_4	x_5	x_4	x_7	x_7	x_7	x_{10}
x_8	x_0	x_0	x_0	x_3	x_4	x_5	x_6	x_7	x_8	x_8	x_{10}
x_9	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_{10}	x_0	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}

$\sqcup$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_0	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
x_1	x_1	x_1	x_2	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_1
x_2	x_2	x_2	x_2	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_2
x_3	x_3	x_9	x_9	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_3
x_4	x_4	x_9	x_9	x_4	x_4	x_6	x_6	x_7	x_8	x_9	x_4
x_5	x_5	x_9	x_9	x_5	x_6	x_5	x_6	x_7	x_8	x_9	x_5
x_6	x_6	x_9	x_9	x_6	x_6	x_6	x_6	x_8	x_8	x_9	x_6
x_7	x_7	x_9	x_9	x_7	x_7	x_7	x_8	x_7	x_8	x_9	x_7
x_8	x_8	x_9	x_9	x_8	x_8	x_8	x_8	x_8	x_8	x_9	x_8
x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9
x_{10}	x_{10}	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}

Thank you for the attention !