Lattices embeddable in three-generated lattices

Gábor Czédli (SSAOS-54, Trojanovice, Sept. 3–9, 2016)

September 5, 2016

Theorem (Czédli, submitted to Acta Sci. Math. (Szeged) in December, 2015)

Every finite lattice is a sublattice of a three-generated finite lattice.

This result strengthens P. Crawley and R. A. Dean's 1959 result by adding **finiteness**.

Remark

Nowadays, with 4 instead of 3 is quite easy. Why?

Proving embeddability into 4-generated.

H. Strietz (1977) (or L. Zádori 1986) plus P. Pudlák and J. Tůma 1980. □

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Definition

A cardinal κ is *inaccessible* if

- $\kappa > \aleph_0$,
- for every cardinal λ, $\lambda < \kappa$ implies that $2^\lambda < \kappa$, and
- for every set I of cardinals, if $|I| < \kappa$ and each member of I is less than κ, then $\sum\{\lambda : \lambda \in I\} < \kappa$.

A cardinal λ will be called *accessible* if there is no inaccessible cardinal κ such that $\kappa \leq \lambda$.

Remark

ZFC has a model in which all cardinal numbers are accessible.

Theorem (Czédli, same paper, December, 2015)

Every ***algebraic*** *lattice with accessible cardinality is a* ***complete*** *sublattice of an appropriate* ***algebraic*** *lattice K such that K, as complete lattice, is generated by three elements.*

Again, this strengthens P. Crawley and R. A. Dean's 1959 results by adding algebraicity and completeness.

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So it suffices to give a complete embedding of $\mathrm{Equ}(A_0)$ into an algebraic lattice 3-generated in complete lattice sense.

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(17-page paper & 20-minute talk) $\Longrightarrow$ only a sketch. □

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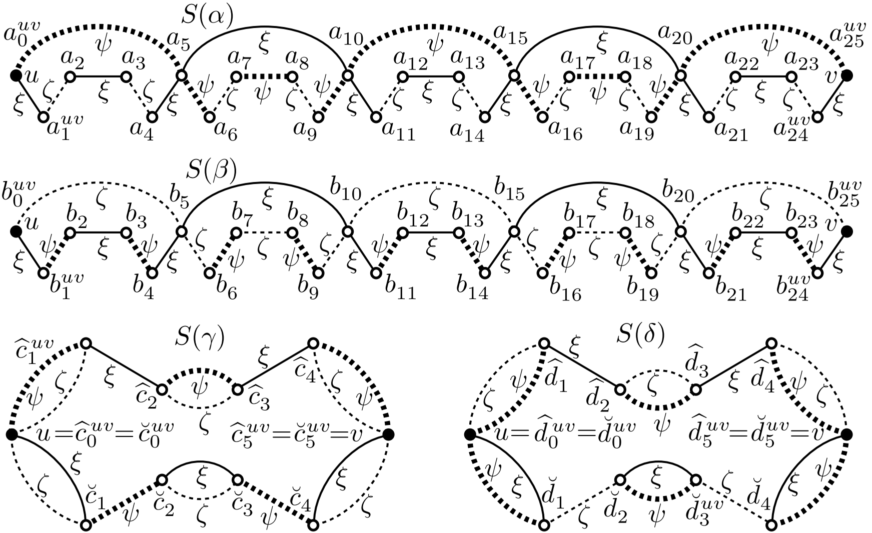
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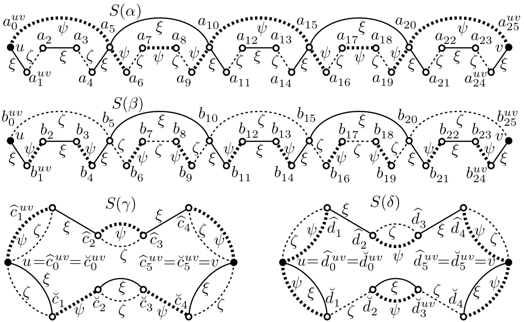
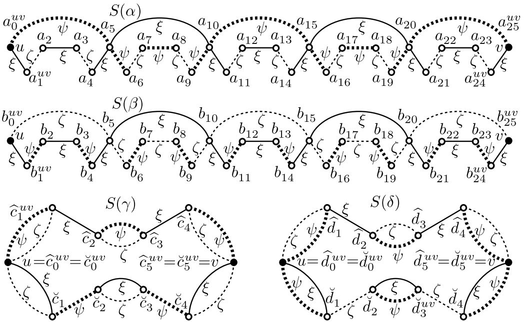


Let $2 \leq |A_0|$ be accessible (finite or infinite), and let $\mathrm{Equ}(A_0) = [\alpha_0, \beta_0, \gamma_0, \delta_0]$, see Zádori (finite case) and Czédli (infinite case). At some point later, which is not detailed now, it will be important that $\alpha_0 \leq \gamma_0 \vee \delta_0$ and $\beta_0 \leq \gamma_0 \vee \delta_0$.



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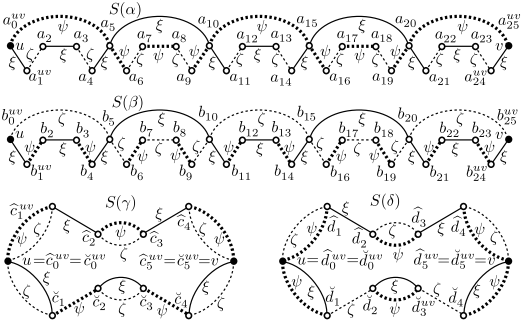
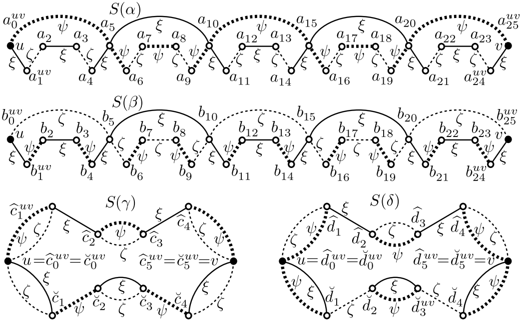


Whenever $u \neq v \in A_0$ and $\langle u, v\rangle \in \alpha_0$, replace the pair $\langle u, v\rangle$ with a (new) copy $S^{uv}(\alpha)$ of $S(\alpha)$. Similarly for $\beta_0, \gamma_0, \delta_0$. In this way, we obtain a larger set, A_1, and a graph structure on A_1. Let $\xi_1 \in \mathrm{Equ}(A_1)$ be the equivalence generated by the ξ-colored edges. We define $\psi_1, \zeta_1 \in \mathrm{Equ}(A_1)$ similarly.



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$$\widehat\alpha = \bigl(\xi \wedge (\psi \vee \zeta)\bigr) \vee \bigl(\psi \wedge (\xi \vee \zeta)\bigr),$$
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we let $\widehat\alpha_1 := \widehat\alpha(\xi_1, \psi_1, \zeta_1)$, $\dots$, $\widehat\delta_1 := \widehat\delta(\xi_1, \psi_1, \zeta_1)$. Finally,

$$\widehat\alpha_2 := \widehat\alpha_1 \wedge (\widehat\gamma_1 \vee \widehat\delta_1),$$
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Definition

Define

$$f\colon \mathrm{Equ}(A_0) \to \mathrm{Equ}(A_1) \text{ by } t(\alpha_0, \ldots, \delta_0) \mapsto t(\widehat{\alpha}_2, \ldots, \widehat{\delta}_2),$$

where t denotes a quaternary term.

This is the required embedding, since $f(\mathrm{Equ}(A_0))$ is a complete sublattice of the (complete) sublattice S generated by $\{\xi_1, \psi_1, \zeta_1\}$ in $\mathrm{Equ}(A_1)$.

I give only two figures; see the paper for more details.

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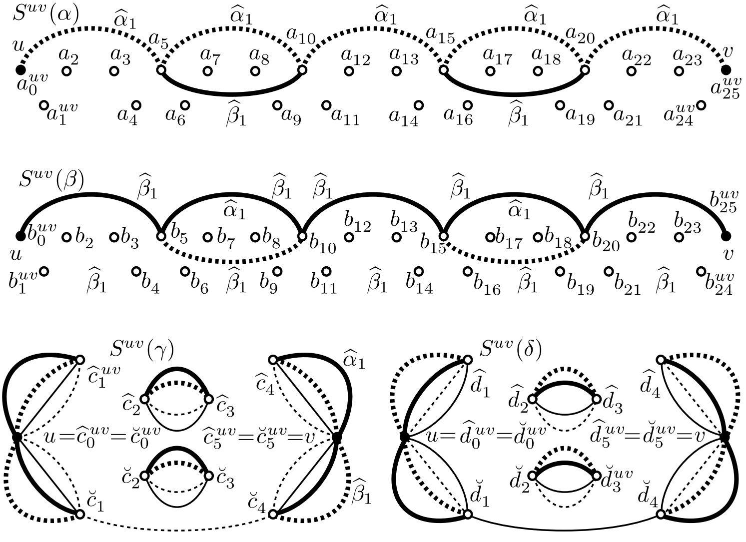

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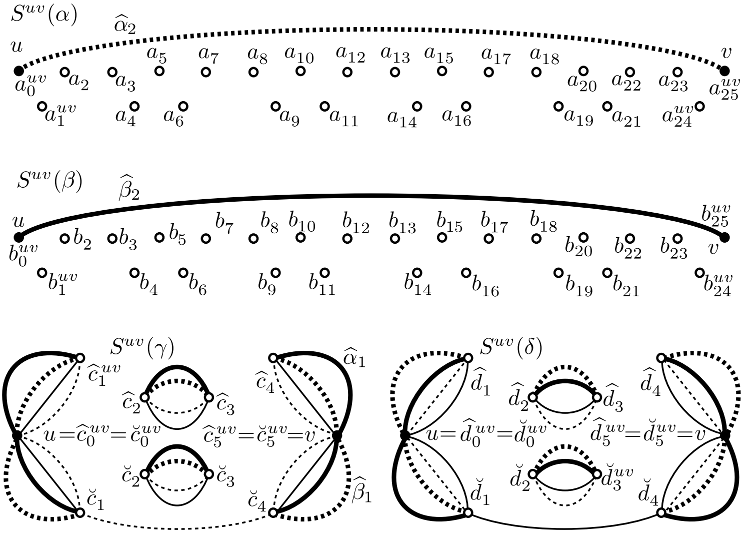

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Based on $\mathrm{Sub}(M) \cong \mathrm{Con}(M)$, András suggested me to find another proof based on Mal'cev (= Mal'tsev) conditions.

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But where to apply my Mal'cev conditions? I applied them for **sets** to obtain an effective algorithm for the word problem of lattices (Periodica Math. Hungar. 1991).
Later, the same technique was used to prove some known properties of free lattices (A visual approach to test lattices, 2009), and this technique was developed further by B. Skublics (Lattice identities and colored graphs connected by test lattices; Novi Sad J. Math. **40** (2010), 109–117).
His paper gives the motivation to choose the auxiliary graphs $S(\alpha)$, ..., $S(\delta)$ as the *Skublics graphs* of rank 5 associated with our auxiliary terms $\widehat{\alpha}$, ..., $\widehat{\delta}$.

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Theorem (Czédli, 2016)

$\mathsf{FL}(3)$ *includes a sublattice* S *such that* $S \cong \mathsf{FL}(\omega)$ *and for the unique dual automorphism* δ *of* $\mathsf{FL}(3)$, *we have that* $\delta(S) = S$.

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http://www.math.u-szeged.hu/~czedli/

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