Notes on quasiorder lattices

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Generating Quord(A)

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Previous results

- H. Strietz (1975), L. Zádori (1986): a four-element generating set for $\mathrm{Equ}(A)$ if A is finite.
- G. Czédli (1996): $\mathrm{Equ}(A)$ is four-generated if there is no inaccessible cardinal m such that $m \leq |A|$.
- I. Chajda and G. Czédli (1996): a six-element generating set for $\mathrm{Quord}(A)$ for all finite and some infinite sets.
- G. Takách (1996): $\mathrm{Quord}(A)$ is six-generated if there is no inaccessible cardinal m such that $m \leq |A|$.
- T. Dolgos (2015): a five-element generating set for $\mathrm{Quord}(A)$ and an eight-element generating set for $\mathrm{Tran}(A)$ for countable sets.

$\mathrm{Quord}(A)$ is five-generated

Theorem (Kulin)

Let A be a set with at least three elements.

1. If there is no inaccessible cardinal m such that $m \leq |A|$, then $\mathrm{Quord}(A)$ has a five-element generating set.
2. If $\aleph_0 \leq |A| \leq 2^{\aleph_0}$, then $\mathrm{Quord}(A)$ has a five-element generating set.

Proof of part (2) of the Theorem

Let $A_0 = \{a_0, b_0, a_1, b_1, a_2, b_2, \dots\}$.

We define five quasiorders on A_0: α_0^0, α_1^0, α_2^0, β^0, and β_*^0.

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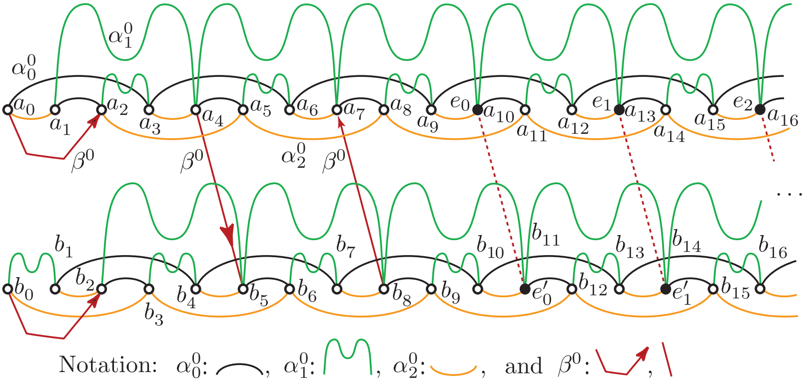

Figure: Quasiorders on A_0

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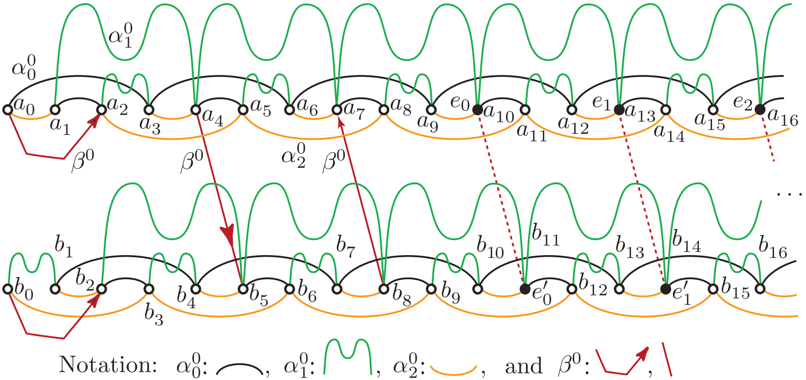

Figure: Quasiorders on A_0

$$\beta_*^0 = (\beta^0)^{-1}$$

Proof of part (2) of the Theorem

Let κ be an arbitrary cardinal such that $\aleph_0 \le \kappa \le 2^{\aleph_0}$. Let $I = \{2, 3, 4, \dots\}$, and $H \subseteq \mathcal{P}(I)$ such that $|H| = \kappa$.

For $U \in H$:

- $A_0(U) = \{a_0(U), b_0(U), a_1(U), b_1(U), a_2(U), b_2(U), \dots\}$,
- we copy the previous edges,
- and we replace the dotted edges with “real” edges (e_i', e_i) for $i \in U$ and (e_i, e_i') for $i \in I \setminus U$.

A new colored graph:
$A = \{A_0(U) : U \in H\}$, the previous edges, and some β-colored directed edges in addition, connecting $A_0(\emptyset)$ with $A_0(U)$ for all $U \in H$.

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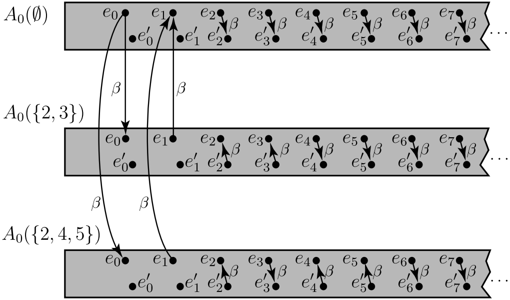

Figure: A part of $\beta \in \mathrm{Quord}(A)$ if $H = \{\emptyset, \{2,3\}, \{2,4,5\}\}$

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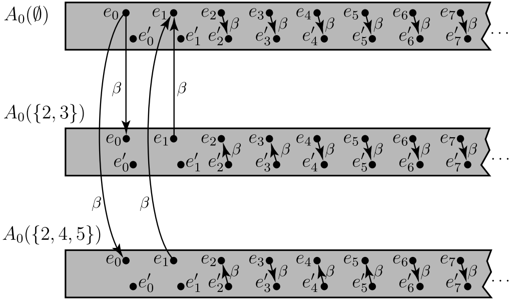

Figure: A part of $\beta \in \mathrm{Quord}(A)$ if $H = \{\emptyset, \{2,3\}, \{2,4,5\}\}$

Five generators: α_0, α_1, α_2, β, and $\beta_* := \beta^{-1}$.

$\mathrm{Quord}(A)$ is four-generated, but not three-generated

Theorem (Czédli)

If A is a set with $|A| \in \{n \in \mathbb{N} : n \geq 11\} \cup \{2, 3, 5, 7, 9, \aleph_0\}$, then $\mathrm{Quord}(A)$ is a four-generated lattice.

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Theorem (Czédli, Kulin)

If $|A| \geq 5$ and there is no inaccessible cardinal m such that $m \leq |A|$, then $\mathrm{Quord}(A)$ is four-generated.

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Theorem (Czédli, Kulin)

If $|A| \geq 58$ and there is no inaccessible cardinal m such that $m \leq |A|$, then $\mathrm{Quord}(A)$ is $(1 + 1 + 2)$-generated.

Generating $\mathrm{Tran}(A)$

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Theorem (Czédli, Kulin)

Let A be a set with at least three elements. If there is no inaccessible cardinal m such that $m \leq |A|$, then $\mathrm{Tran}(A)$ has a six-element generating set.

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