On the lattice of congruence lattices of algebras on a finite set

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Outline

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Compatible relations — congruences, quasiorders

$\langle A, F \rangle$ universal algebra
compatible (invariant) relation $q \subseteq A \times A$:
For each $f \in F$ (n-ary) we have $f \triangleright q$ (f preserves q), i.e.

$$(a_1, b_1), \ldots, (a_n, b_n) \in q \implies (f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in q\,.$$

$\mathrm{Con}(A, F)$ compatible equivalence relations $=$ *congruences*

Remark

$(\mathrm{Con}\langle A, F \rangle, \subseteq)$ is a lattice and it is a complete sublattice of the lattice $(\mathrm{Eq}(A), \subseteq)$ of all equivalence relations on A.

Problem

Describe the lattice
$\mathcal{E} := (\{\mathrm{Con}\langle A, F \rangle \mid F \textit{ set of operations on } A\}, \subseteq)$.
(in particular atoms and coatoms)

Reduction to (mono)unary algebras

$H :=$ unary polynomial operations of $\langle A, F\rangle$ (i.e. $H = \langle F \cup C\rangle^{(1)}$).
It is well-known that

$$\mathrm{Con}\langle A, F\rangle = \mathrm{Con}\langle A, H\rangle$$
$$\mathrm{Con}\langle A, H\rangle = \bigcap_{f\in H} \mathrm{Con}\langle A, f\rangle.$$

Thus $\mathcal{E} = (\{\mathrm{Con}\langle A, H\rangle \mid H \leq A^A\}, \subseteq)$.

Description of $\mathcal{E}$: look for $\vee$- and $\wedge$-irreducible elements

Remark: End $-$ Con is a Galois connection (induced by $\triangleright$).

$$L \subseteq \mathrm{Con}\langle A, H\rangle \iff \mathrm{End}\, L \supseteq H$$

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Coatoms of $\mathcal{E}$

∧-irreducibles, in particular coatoms, are of the form $\mathrm{Con}(A, f)$ for some nontrivial $f \in A^A$.

Which f yield coatoms?

Theorem

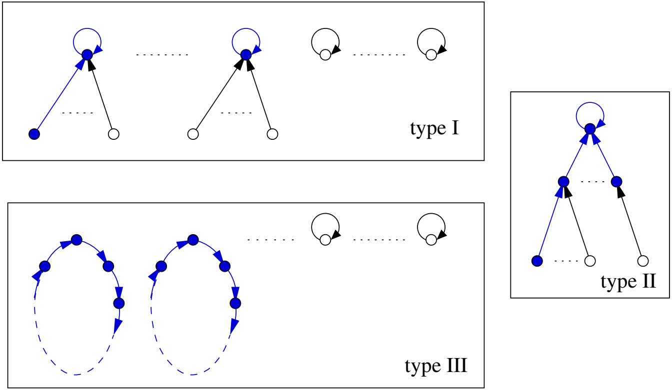
The coatoms of $\mathcal{E}$ are exactly the congruence lattices of the form $\mathrm{Con}(A, f)$ where $f \in A^A$ is nontrivial (i.e., not constant and not the identity) and satisfies

(I) $f^2 = f$, *or*

(II) f^2 *is a constant, say* 0, *and* $|[0]_{\ker f}| \geq 3$, *or*

(III) $f^p = \mathrm{id}_A$ *for some prime* p *such that the permutation* f *has at least two cycles of length* p.

The three types of coatoms of $\mathcal{E}$



∧-irreducibles: Results from 2016

The ∧-irreducible elements $\mathrm{Con}(A, f)$ of can be described explicitly for the following types of functions f:

- permutations
  (then the principle filter $[\mathrm{Con}(A, f)\rangle_{\mathcal{E}}$ is a chain of ∧-irreducible elements)
- acyclic functions

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$\vee$-irreducibles and atoms

Each $\vee$-irreducible element $L = \mathrm{Con}(A, H)$ in $\mathcal{E}$ is of the form

$$E_\varkappa = \mathrm{Con}(A, \mathrm{End}\,\varkappa) \text{ for some } \varkappa \in \mathrm{Con}(A).$$

Question: Which $\varkappa$ yield $\vee$-irreducibles?

Answer: Every nontrivial $\varkappa$

Theorem

The completely $\vee$-irreducibles of $\mathcal{E}$ are exactly the congruence lattices of the form

$$E_\varkappa = \mathrm{Con}(A, \mathrm{End}\,\varkappa) = \{\Delta, \varkappa, \nabla\}$$

where $\varkappa \in \mathrm{Eq}(A) \setminus \{\Delta, \nabla\}$ is an arbitrary equivalence relation. Moreover, each $\vee$-irreducible is an atom in $\mathcal{E}$, i.e.
the lattice $\mathcal{E}$ is atomistic.

Constructions via atoms

$L \in \mathcal{E}$

$\mathrm{At}(L) :=$ atoms $E_\varkappa = \{\Delta, \varkappa, \nabla\}$ of $\mathcal{E}$ contained in $L \in \mathcal{E}$

$\mathcal{E}$ atomistic $\implies L = \bigvee \mathrm{At}(L) = \bigcup \mathrm{At}(L) = \{\varkappa \mid E_\varkappa \subseteq L\}$
(in general, for arbitrary L: $\bigvee \neq \bigcup$!)

Question: When a set of atoms is the set $\mathrm{At}(L)$ for some $L \in \mathcal{E}$?

Given $E \subseteq \mathrm{Eq}(A)$, find $L = \mathrm{Con}(A, \mathrm{End}\, E) = \bigvee_{\varkappa \in E} E_\varkappa$,
i.e., find $\mathrm{At}(L)$.
Formally we put $[E] := \mathrm{At}(L) \cup \{\Delta, \nabla\}$

What is the closure operator $E \mapsto [E]$?

The closure operator for atoms

The closure $[E]$ is coincides with the Galois closure of a known Galois connection (namely, $\mathsf{Pol} - \mathsf{Inv}$ or $\mathsf{End} - \mathsf{Inv}$) studied in the framework of a *General Galois theory for operations and relations* (for clones of operations and relations)

For $E \subseteq \mathrm{Eq}(A)$ we have:

$$\begin{aligned} [E] &= \mathrm{Con}\,\mathrm{End}\,E \\ &= \mathrm{Eq}(A) \cap \mathrm{Inv}\,\mathrm{End}\,E \\ (&= \mathrm{Eq}(A) \cap \mathrm{Inv}\,\mathrm{Pol}\,E\ ) \\ &= \mathrm{Eq}(A) \cap [E]_{\mathrm{wKC}} \text{ (weak Krasner clone)} \\ (&= \mathrm{Eq}(A) \cap [E]_{\mathrm{RC}} \text{ (relational clone)}\ ) \end{aligned}$$

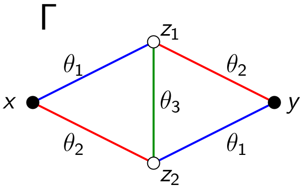
Constructive approach to the closure for atoms

H. Werner (1974, *Which partition lattices are congruence lattices?*) described the closure operator

$$E \mapsto \mathrm{Con}(A, \mathrm{End}\, E)$$

via so-called *graphical compositions*: $E = \mathrm{Con}(A, \mathrm{End}\, E)$ iff E is closed under graphical compositions.

Example:



Graphical composition corresponding to the coloured graph Γ

$$f_\Gamma(\theta_1, \theta_2, \theta_3) := \{(x, y) \mid \exists z_1, z_2 : (x, z_1) \in \theta_1 \wedge (x, z_2) \in \theta_2 \wedge (z_1, y) \in \theta_2 \wedge (z_2, y) \in \theta_1 \wedge (z_1, z_2) \in \theta_3\}$$

Outline

Notions and notations

$\wedge$-irreducibles and coatoms of $\mathcal{E}$

$\vee$-irreducibles (in particular atoms) of $\mathcal{E}$

Lattice-theoretical properties of $\mathcal{E}$

Intersection of coatoms and join of atoms

Proposition

There are two or three coatoms in the lattice $\mathcal{E}$ whose meet is $\mathbf{0}_{\mathcal{E}}$. More precisely, for $|A| > 5$, there are two coatoms $\mathrm{Con}(A, f)$ and $\mathrm{Con}(A, g)$ such that $\mathrm{Con}(A, f) \cap \mathrm{Con}(A, g) = \{\Delta, \nabla\}$. For $|A| \leq 5$, three coatoms are necessary (and sufficient) for this property.

Proposition

There are three atoms in $\mathcal{E}$ whose join is $\mathbf{1}_{\mathcal{E}}$. More precisely, there are three equivalence relations $\varkappa_1, \varkappa_2, \varkappa_3$ such that $E_{\varkappa_1} \vee E_{\varkappa_2} \vee E_{\varkappa_3} = \mathrm{Eq}(A)$.

Tolerance simplicity

Recall: a lattice is *tolerance-simple* if it has no non-trivial tolerances (compatible reflexive and symmetric relations).

Theorem

For $|A| \geq 4$, the lattice $\mathcal{E}$ is tolerance-simple.

Sketch of the proof.

- Tolerance simplicity follows from the property that $T(\mathbf{0}_{\mathcal{E}}, E) = T(\mathbf{0}_{\mathcal{E}}, E')$ for all atoms $E, E' \in \mathcal{E}$.
- For each nontrivial $\varkappa \in \mathrm{Eq}(A)$ there exists $(a, b) \in \varkappa$ such that $T(\mathbf{0}_{\mathcal{E}}, E_{\varkappa}) = T(\mathbf{0}_{\mathcal{E}}, E_{[a,b]})$.
- $T(\mathbf{0}_{\mathcal{E}}, E_{[a,b]}) = T(\mathbf{0}_{\mathcal{E}}, E_{[a',b']})$ for all $(a, b), (a', b') \in A^2 \setminus \Delta$.

For more details ask Sandor $\square$

Lattice properties which fail to hold

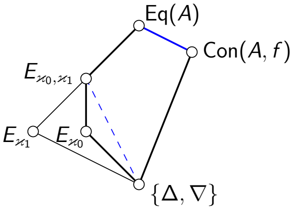
Proposition

For $|A| \geq 4$: the lattice $\mathcal{E}$ has none of the following properties: 0*-modular,* 1*-modular, lower semimodular.*

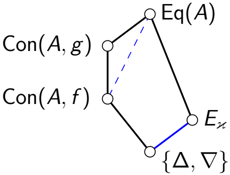
For $|A| \geq 8$: $\mathcal{E}$ is not upper semimodular.

$\varkappa_0 = [0,1]$, $\varkappa_1 = [0,1,2]$,

$$fx := \begin{cases} 3 & \text{if } x = 0, \\ x & \text{otherwise.} \end{cases}$$



$f = (0123)(0'1'2'3')$,
$g := f^2 = (02)(13)(0'2')(1'3')$,
$\varkappa = [0,0']$



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