Special elements in lattices
Weak congruences
Ω-algebras

Branimir Šešelja
Department of Mathematics and Informatics
Faculty of Sciences, University of Novi Sad

SSAOS 2016

Trojanovice, *September 6, 2016*

Special elements in a lattice

Special elements in a lattice

Ore (1935, 1942), Birkhoff (1940), Grätzer (1959, 1978), Grätzer and Schmidt (1961), Hashimoto and Kinugawa (1963), Reilley (1984)...

Special elements in a lattice

Ore (1935, 1942), Birkhoff (1940), Grätzer (1959, 1978), Grätzer and Schmidt (1961), Hashimoto and Kinugawa (1963), Reilley (1984)...

An element a of a lattice L is **distributive** if for every $x, y \in L$,

$$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y).$$

Special elements in a lattice

Ore (1935, 1942), Birkhoff (1940), Grätzer (1959, 1978), Grätzer and Schmidt (1961), Hashimoto and Kinugawa (1963), Reilley (1984)...

An element a of a lattice L is **distributive** if for every $x, y \in L$,

$$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y).$$

Dually, a is **codistributive** if

$$a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y).$$

Special elements in a lattice

Ore (1935, 1942), Birkhoff (1940), Grätzer (1959, 1978), Grätzer and Schmidt (1961), Hashimoto and Kinugawa (1963), Reilley (1984)...

An element a of a lattice L is **distributive** if for every $x, y \in L$,

$$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y).$$

Dually, a is **codistributive** if

$$a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y).$$

An element a of a lattice L is **neutral** if for all $x, y \in L$,

$$(a \vee x) \wedge (a \vee y) \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) \vee (x \wedge y).$$

Special elements in a lattice

Ore (1935, 1942), Birkhoff (1940), Grätzer (1959, 1978), Grätzer and Schmidt (1961), Hashimoto and Kinugawa (1963), Reilley (1984)...

An element a of a lattice L is **distributive** if for every $x, y \in L$,

$$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y).$$

Dually, a is **codistributive** if

$$a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y).$$

An element a of a lattice L is **neutral** if for all $x, y \in L$,

$$(a \vee x) \wedge (a \vee y) \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) \vee (x \wedge y).$$

An element $a \in L$ is **cancellable**, if
from $x \wedge a = y \wedge a$ and $x \vee a = y \vee a$ it follows that $x = y$.

An element a is neutral if and only if it is distributive, codistributive and cancellable.

An element a is neutral if and only if it is distributive, codistributive and cancellable.

Element a is **standard** if for all $x, y \in L$,

$$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y).$$

An element a is neutral if and only if it is distributive, codistributive and cancellable.

Element a is **standard** if for all $x, y \in L$,

$$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y).$$

An element satisfying the dual law

$$x \vee (a \wedge y) = (x \vee a) \wedge (x \vee y),$$

is called **costandard**.

An element a is neutral if and only if it is distributive, codistributive and cancellable.

Element a is **standard** if for all $x, y \in L$,

$$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y).$$

An element satisfying the dual law

$$x \vee (a \wedge y) = (x \vee a) \wedge (x \vee y),$$

is called **costandard**.
a is **modular** if for all $x, y \in L$,

$$\text{from } a \leqslant y \text{ it follows that } a \vee (x \wedge y) = (a \vee x) \wedge y.$$

An element a is neutral if and only if it is distributive, codistributive and cancellable.

Element a is **standard** if for all $x, y \in L$,

$$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y).$$

An element satisfying the dual law

$$x \vee (a \wedge y) = (x \vee a) \wedge (x \vee y),$$

is called **costandard**.
a is **modular** if for all $x, y \in L$,

$$\text{from } a \leqslant y \text{ it follows that } a \vee (x \wedge y) = (a \vee x) \wedge y.$$

a is **s-modular** if for all $x, y \in L$,

$$\text{from } x \leqslant y \text{ it follows that } x \vee (a \wedge y) = (x \vee a) \wedge y.$$

Theorem

If a is an element of a lattice L, then the following conditions are equivalent:
(1) a is a distributive element in L;
(2) the function $n_a : x \mapsto a \vee x$ is a homomorphism from L onto the principal filter $\uparrow a$;
(3) the relation θ_a on L, defined by

$$x\,\theta_a y \text{ if and only if } a \vee x = a \vee y,$$

is a congruence.

Theorem

Let a be an element from a lattice L. The following conditions are equivalent:
(i) a is codistributive;
(ii) the mapping $m_a : L \longrightarrow \downarrow a$ defined by $m_a(x) = a \wedge x$ is a lattice homomorphism;
(iii) binary relation θ_a defined by:

$$(x, y) \in \theta_a \text{ if and only if } a \wedge x = a \wedge y,$$

is a congruence relation on L.

Theorem

Let L be a lattice and $a \in L$. The following conditions are equivalent:
(i) a is neutral;
(ii) a is distributive, codistributive and cancellable;
(iii) a is standard and costandard;
(iv) The mappings m_a and n_a are homomorphisms and the mapping $x \mapsto (x \wedge a, x \vee a)$ is an embedding from L to $\downarrow a \times \uparrow a$;
(v) For all $x, y \in L$, the sublattice generated by $\{x, y, a\}$ is distributive.

The **center** of a bounded lattice L is the set of all elements from L which are neutral and have complements.

The **center** of a bounded lattice L is the set of all elements from L which are neutral and have complements.

Theorem

An element a belongs to the center of a bounded lattice L if and only if

$$L \cong \downarrow a \times \uparrow a \quad \textit{under} \quad x \mapsto (x \wedge a, x \vee a).$$

The **center** of a bounded lattice L is the set of all elements from L which are neutral and have complements.

Theorem

An element a belongs to the center of a bounded lattice L if and only if

$$L \cong \downarrow a \times \uparrow a \quad \textit{under} \quad x \mapsto (x \wedge a, x \vee a).$$

If a is a codistributive element and if the congruence block $[x]_{\theta_a}$ of an $x \in L$ has the top element, then we denote it by $\overline{x}$.

The **center** of a bounded lattice L is the set of all elements from L which are neutral and have complements.

Theorem

An element a belongs to the center of a bounded lattice L if and only if

$$L \cong \downarrow a \times \uparrow a \quad \textit{under} \quad x \mapsto (x \wedge a, x \vee a).$$

If a is a codistributive element and if the congruence block $[x]_{\theta_a}$ of an $x \in L$ has the top element, then we denote it by $\overline{x}$.

Proposition

If a is in the center of a lattice L, then every block of the congruence induced by m_a (n_a) has the top (the bottom) element.

Proposition

If a is a codistributive element of the lattice L, and the top elements in the congruence blocks induced by m_a exist, then the following are equivalent:
(i) *a is modular and cancellable;*
(ii) *for every $x \in \downarrow a$, the map $y \mapsto y \vee a$ is an isomorphism from the interval $[x, \overline{x}]$ onto the interval $[a, \overline{x} \vee a]$.*

Element a of a bounded lattice is **exceptional** if it is neutral and the blocks of the congruence θ_a induced by m_a have top elements which form a sublattice M_a of L.

Element a of a bounded lattice is **exceptional** if it is neutral and the blocks of the congruence θ_a induced by m_a have top elements which form a sublattice M_a of L.

Proposition

A codistributive element a of a lattice L is exceptional if and only if the following two statements are true:
(i) *for every* $x \in \downarrow a$, $\quad [x, \overline{x}] \cong [a, \overline{x} \vee a]$ *under* $y \mapsto y \vee a$, *and*
(ii) *the mapping* $x \mapsto \overline{x} \vee a$ *is a homomorphism from* $\downarrow a$ *to* $\uparrow a$.

An element a of a lattice L is said to be **infinitely distributive** if for each family $\{x_i \mid i \in I\} \subseteq L$

$$a \vee (\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I} (a \vee x_i).$$

An element a of a lattice L is said to be **infinitely distributive** if for each family $\{x_i \mid i \in I\} \subseteq L$

$$a \vee (\bigwedge_{i\in I} x_i) = \bigwedge_{i\in I} (a \vee x_i).$$

An element satisfying the dual law is called **infinitely codistributive**.

An element a of a lattice L is said to be **infinitely distributive** if for each family $\{x_i \mid i \in I\} \subseteq L$

$$a \vee (\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I} (a \vee x_i).$$

An element satisfying the dual law is called **infinitely codistributive**.

Proposition

The following conditions are equivalent for an element $a \in L$:
(i) a is infinitely distributive;
(ii) the mapping $n_a : L \longrightarrow \uparrow a$ defined by $n_a(x) = a \vee x$ is a complete homomorphism.
(iii) Binary relation σ_a defined by: $(x, y) \in \sigma_a$ if and only if $a \vee x = a \vee y$ is a complete congruence on L.

Proposition

An element a of L is infinitely distributive if and only if for every $b \in \uparrow a$*, the family* $\{x \in L \mid a \vee x \geqslant b\}$ *has the bottom element.*

Proposition

An element a of L is infinitely distributive if and only if for every $b \in \uparrow a$, the family $\{x \in L \mid a \vee x \geqslant b\}$ has the bottom element.

Proposition

An element a of L is infinitely distributive if and only if for every $b \in \uparrow a$, the family $\{x \in L \mid a \vee x = b\}$ has the bottom element and a is a modular element of L.

Proposition

An element a of L is infinitely distributive if and only if for every $b \in \uparrow a$, the family $\{x \in L \mid a \vee x \geqslant b\}$ has the bottom element.

Proposition

An element a of L is infinitely distributive if and only if for every $b \in \uparrow a$, the family $\{x \in L \mid a \vee x = b\}$ has the bottom element and a is a modular element of L.

Theorem

The following statements are equivalent for an element a of L:
(i) a is cancellable and $\downarrow a \cong \uparrow a$, under $x \mapsto \overline{x} \vee a$;
(ii) a is an exceptional infinitely distributive element, and the set of all bottom elements $\{\underline{x} \mid x \in L\}$ is equal to the set M_a of all top elements.

Theorem

If a is a neutral element of the lattice L, then an arbitrary lattice identity is satisfied on L if and only if this identity holds on $\downarrow a$ and on $\uparrow a$.

Theorem

(i) (L. Libkin 1995) *In an atomistic algebraic lattice an element is neutral if and only if it is distributive and codistributive.*

(ii) (S. Radeleczki 2000) *Every costandard element in an atomistic lattice is neutral.*

(iii) (B. Šešelja, A. Tepavčević 2008) *A codistributive element s in an atomistic algebraic lattice L has a complement s' which is distributive. In addition,*

(a) *The kernels of the homomorphisms $x \mapsto x \wedge s$ from L to $\downarrow s$ and $x \mapsto x \vee s'$ from L to $\uparrow s'$ coincide,*

(b) $\uparrow s' \cong \downarrow s$, *under* $x \mapsto x \wedge s$.

An element d of a lattice L is called a **join-semidistributive** if

$$d \vee x = d \vee y \Rightarrow d \vee (x \wedge y) = d \vee x$$

holds for all x and y in L.

Weak equivalences on a set

Weak equivalences on a set

In 1946, O. Borůvka, inspired by a paper of O. Ore, introduced **equivalences in a set**, as symmetric and transitive relations on a set.

Weak equivalences on a set

In 1946, O. Borůvka, inspired by a paper of O. Ore, introduced **equivalences in a set**, as symmetric and transitive relations on a set.
These are in a bijection with **partitions in a set**.

Weak equivalences on a set

In 1946, O. Borůvka, inspired by a paper of O. Ore, introduced **equivalences in a set**, as symmetric and transitive relations on a set.
These are in a bijection with **partitions in a set**.
Detailed description of the lattice of partitions in a set was done by H. Draškovičová in 1970.

Weak equivalences on a set

In 1946, O. Borůvka, inspired by a paper of O. Ore, introduced **equivalences in a set**, as symmetric and transitive relations on a set.
These are in a bijection with **partitions in a set**.
Detailed description of the lattice of partitions in a set was done by H. Draškovičová in 1970.

For an $(s)(t)$ relation ρ on a nonempty set A we use the name **weak equivalence** on A. Each weak equivalence (except the empty relation, which is also a weak equivalence on A) is an (ordinary) equivalence relation on a subset $A\rho$ of A:

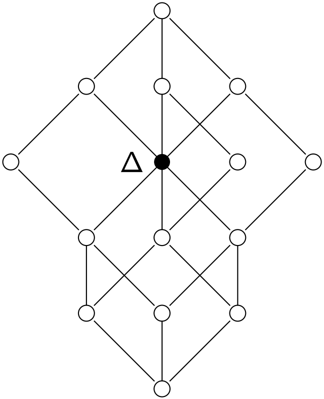
$$A\rho = \{x \in A \mid x\rho x\}.$$

Proposition (Draškovičová, 1970)

The collection $\mathcal{E}w(A)$ of all weak equivalences on A is an algebraic lattice under inclusion. This lattice is upper-continuous and semimodular.

Proposition (Draškovičová, 1970)

The collection $\mathcal{E}w(A)$ of all weak equivalences on A is an algebraic lattice under inclusion. This lattice is upper-continuous and semimodular.



The lattice of weak equivalences of a three-element set

Weak congruence relations

Weak congruence relations

F. Šik, together with his Ph.D. student T.D. Mai (1974), was first to investigate compatible $(s)(t)$ relations on algebras, mainly on Ω-groups.

Weak congruence relations

F. Šik, together with his Ph.D. student T.D. Mai (1974), was first to investigate compatible $(s)(t)$ relations on algebras, mainly on Ω-groups.

They called these relations congruences **in** algebras, and investigated lattices of such relations associated to group-like algebras.

Weak congruence relations

F. Šik, together with his Ph.D. student T.D. Mai (1974), was first to investigate compatible $(s)(t)$ relations on algebras, mainly on Ω-groups.
They called these relations congruences **in** algebras, and investigated lattices of such relations associated to group-like algebras.

In 1988. B. Šešelja and G. Vojvodić introduced the notion **weak congruence** on an algebra, and described some basic properties of the corresponding lattice.

Definition

Definition

A **weak congruence** on an algebra $\mathcal{A}$ is a symmetric and transitive subuniverse of $\mathcal{A}^2$.

Definition

A **weak congruence** on an algebra $\mathcal{A}$ is a symmetric and transitive subuniverse of $\mathcal{A}^2$.
Equivalently, it is a symmetric, transitive and compatible relation θ on an algebra $\mathcal{A}$, hence fulfilling the **weak reflexivity**:

Definition

A **weak congruence** on an algebra $\mathcal{A}$ is a symmetric and transitive subuniverse of $\mathcal{A}^2$.
Equivalently, it is a symmetric, transitive and compatible relation θ on an algebra $\mathcal{A}$, hence fulfilling the **weak reflexivity**:

For every nullary operation c in the language of $\mathcal{A}$, $c\theta c$.

Definition

A **weak congruence** on an algebra $\mathcal{A}$ is a symmetric and transitive subuniverse of $\mathcal{A}^2$.
Equivalently, it is a symmetric, transitive and compatible relation θ on an algebra $\mathcal{A}$, hence fulfilling the **weak reflexivity**:

For every nullary operation c in the language of $\mathcal{A}$, $c\theta c$.

By the definition, if $\mathcal{A}$ has no fundamental nullary operations, then the empty set is also a weak congruence on this algebra.

Definition

A **weak congruence** on an algebra $\mathcal{A}$ is a symmetric and transitive subuniverse of $\mathcal{A}^2$.
Equivalently, it is a symmetric, transitive and compatible relation θ on an algebra $\mathcal{A}$, hence fulfilling the **weak reflexivity**:

For every nullary operation c in the language of $\mathcal{A}$, $c\theta c$.

By the definition, if $\mathcal{A}$ has no fundamental nullary operations, then the empty set is also a weak congruence on this algebra.

Clearly, every congruence on a subalgebra of $\mathcal{A}$ is a weak congruence on $\mathcal{A}$, and vice versa, every nonempty weak congruence θ on $\mathcal{A}$ is a congruence on a subalgebra $\mathcal{B}_\theta$ of $\mathcal{A}$, where $B_\theta := \{x \in A \mid x\,\theta\,x\}$.

The weak congruences on $\mathcal{A}$ form an algebraic lattice under inclusion, denoted by $\mathrm{Con_w}(\mathcal{A})$.

The weak congruences on $\mathcal{A}$ form an algebraic lattice under inclusion, denoted by $\mathrm{Con_w}(\mathcal{A})$.

The congruence lattice $\mathrm{Con}(\mathcal{A})$ *of* $\mathcal{A}$ *is a principal filter in* $\mathrm{Con_w}(\mathcal{A})$, *generated by the diagonal relation* Δ *of* A.

The weak congruences on $\mathcal{A}$ form an algebraic lattice under inclusion, denoted by $\mathrm{Con_w}(\mathcal{A})$.

The congruence lattice $\mathrm{Con}(\mathcal{A})$ of $\mathcal{A}$ is a principal filter in $\mathrm{Con_w}(\mathcal{A})$, generated by the diagonal relation Δ of A.

The congruence lattice of any subalgebra of $\mathcal{A}$ is an interval sublattice of $\mathrm{Con_w}(\mathcal{A})$.

The weak congruences on $\mathcal{A}$ form an algebraic lattice under inclusion, denoted by $\mathrm{Con_w}(\mathcal{A})$.

The congruence lattice $\mathrm{Con}(\mathcal{A})$ of $\mathcal{A}$ is a principal filter in $\mathrm{Con_w}(\mathcal{A})$, generated by the diagonal relation Δ of A.

The congruence lattice of any subalgebra of $\mathcal{A}$ is an interval sublattice of $\mathrm{Con_w}(\mathcal{A})$.

The subalgebra lattice $\mathrm{Sub}(\mathcal{A})$ is isomorphic to the principal ideal generated by Δ, by sending each weak congruence θ contained in Δ to its domain.

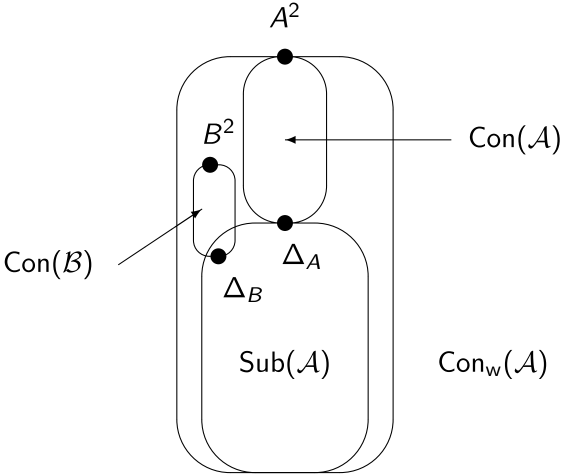
The weak congruences on $\mathcal{A}$ form an algebraic lattice under inclusion, denoted by $\mathrm{Con_w}(\mathcal{A})$.

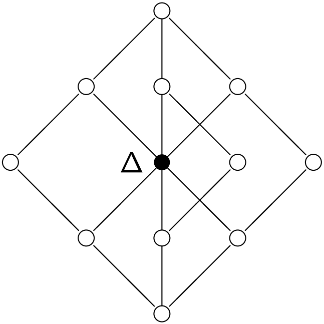
The congruence lattice $\mathrm{Con}(\mathcal{A})$ of $\mathcal{A}$ is a principal filter in $\mathrm{Con_w}(\mathcal{A})$, generated by the diagonal relation Δ of A.

The congruence lattice of any subalgebra of $\mathcal{A}$ is an interval sublattice of $\mathrm{Con_w}(\mathcal{A})$.

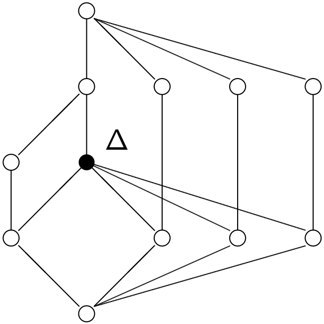
The subalgebra lattice $\mathrm{Sub}(\mathcal{A})$ is isomorphic to the principal ideal generated by Δ, by sending each weak congruence θ contained in Δ to its domain.

Therefore, both the subalgebra lattice and the congruence lattice of an algebra may be recovered and investigated within a single algebraic lattice.

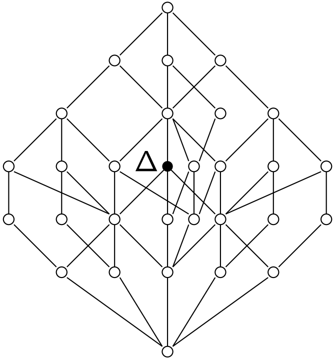

A^2
B^2
$\mathrm{Con}(\mathcal{A})$
$\mathrm{Con}(\mathcal{B})$
Δ_A
Δ_B
$\mathrm{Sub}(\mathcal{A})$
$\mathrm{Con_w}(\mathcal{A})$



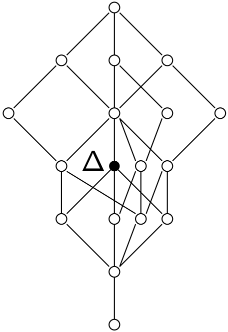
$\mathrm{Con_w}(\mathcal{K})$



$\mathrm{Con_w}(\mathcal{S}_3)$



a) *dihedral group of order 8*



b) *quaternion group*

Congruence Intersection Property, CIP

Congruence Intersection Property, CIP

If ρ is a congruence on a subalgebra of $\mathcal{A}$, then let

$$\rho_{\mathcal{A}} := \bigcap(\theta \in \mathrm{Con}\mathcal{A} \mid \rho \subseteq \theta).$$

Congruence Intersection Property, CIP

If ρ is a congruence on a subalgebra of $\mathcal{A}$, then let

$$\rho_{\mathcal{A}} := \bigcap(\theta \in \mathrm{Con}\mathcal{A} \mid \rho \subseteq \theta).$$

In the lattice of weak congruences, $\rho_{\mathcal{A}} = \rho \vee \Delta$.

Congruence Intersection Property, CIP

If ρ is a congruence on a subalgebra of $\mathcal{A}$, then let

$$\rho_{\mathcal{A}} := \bigcap(\theta \in \mathrm{Con}\mathcal{A} \mid \rho \subseteq \theta).$$

In the lattice of weak congruences, $\rho_{\mathcal{A}} = \rho \vee \Delta$.
$\mathcal{A}$ is said to have the **congruence intersection property** (**CIP**) if for any $\rho \in \mathrm{Con}\,\mathcal{B}$, $\theta \in \mathrm{Con}\,\mathcal{C}$, $\mathcal{B}, \mathcal{C} \in \mathrm{Sub}\,\mathcal{A}$,

$$(\rho \cap \theta)_{\mathcal{A}} = \rho_{\mathcal{A}} \cap \theta_{\mathcal{A}}.$$

Congruence Intersection Property, CIP

If ρ is a congruence on a subalgebra of $\mathcal{A}$, then let

$$\rho_{\mathcal{A}} := \bigcap(\theta \in \mathrm{Con}\mathcal{A} \mid \rho \subseteq \theta).$$

In the lattice of weak congruences, $\rho_{\mathcal{A}} = \rho \vee \Delta$.
$\mathcal{A}$ is said to have the **congruence intersection property** (**CIP**) if for any $\rho \in \mathrm{Con}\,\mathcal{B}$, $\theta \in \mathrm{Con}\,\mathcal{C}$, $\mathcal{B}, \mathcal{C} \in \mathrm{Sub}\,\mathcal{A}$,

$$(\rho \cap \theta)_{\mathcal{A}} = \rho_{\mathcal{A}} \cap \theta_{\mathcal{A}}.$$

In lattice terms, an algebra has the CIP if and only if

$$\Delta \vee (\rho \wedge \theta) = (\Delta \vee \rho) \wedge (\Delta \vee \theta).$$

Congruence Intersection Property, CIP

If ρ is a congruence on a subalgebra of $\mathcal{A}$, then let

$$\rho_{\mathcal{A}} := \bigcap(\theta \in \mathrm{Con}\mathcal{A} \mid \rho \subseteq \theta).$$

In the lattice of weak congruences, $\rho_{\mathcal{A}} = \rho \vee \Delta$.
$\mathcal{A}$ is said to have the **congruence intersection property** (**CIP**) if for any $\rho \in \mathrm{Con}\,\mathcal{B}$, $\theta \in \mathrm{Con}\,\mathcal{C}$, $\mathcal{B}, \mathcal{C} \in \mathrm{Sub}\,\mathcal{A}$,

$$(\rho \cap \theta)_{\mathcal{A}} = \rho_{\mathcal{A}} \cap \theta_{\mathcal{A}}.$$

In lattice terms, an algebra has the CIP if and only if

$$\Delta \vee (\rho \wedge \theta) = (\Delta \vee \rho) \wedge (\Delta \vee \theta).$$

Hence, $\mathcal{A}$ *has the CIP if and only if* Δ *is a distributive element of the lattice* $Cw\mathcal{A}$*,*

Congruence Intersection Property, CIP

If ρ is a congruence on a subalgebra of $\mathcal{A}$, then let

$$\rho_{\mathcal{A}} := \bigcap(\theta \in \mathrm{Con}\mathcal{A} \mid \rho \subseteq \theta).$$

In the lattice of weak congruences, $\rho_{\mathcal{A}} = \rho \vee \Delta$.
$\mathcal{A}$ is said to have the **congruence intersection property** (**CIP**) if for any $\rho \in \mathrm{Con}\,\mathcal{B}$, $\theta \in \mathrm{Con}\,\mathcal{C}$, $\mathcal{B}, \mathcal{C} \in \mathrm{Sub}\,\mathcal{A}$,

$$(\rho \cap \theta)_{\mathcal{A}} = \rho_{\mathcal{A}} \cap \theta_{\mathcal{A}}.$$

In lattice terms, an algebra has the CIP if and only if

$$\Delta \vee (\rho \wedge \theta) = (\Delta \vee \rho) \wedge (\Delta \vee \theta).$$

Hence, *$\mathcal{A}$ has the CIP if and only if Δ is a distributive element of the lattice $Cw\mathcal{A}$, if and only if $n_{\Delta} : \rho \mapsto \rho \vee \Delta$ is a homomorphism from* $\mathrm{Con_w}(\mathcal{A})$ *onto* $\uparrow\Delta$.

A special case of the CIP is the **weak congruence intersection property** (wCIP).

A special case of the CIP is the **weak** congruence intersection property (wCIP).
An algebra $\mathcal{A}$ satisfies the wCIP if for any congruence ρ on a subalgebra of $\mathcal{A}$ and for any congruence θ on $\mathcal{A}$,

$$(\rho \cap \theta)_{\mathcal{A}} = \rho_{\mathcal{A}} \cap \theta.$$

A special case of the CIP is the **weak** congruence intersection property (wCIP).
An algebra $\mathcal{A}$ satisfies the wCIP if for any congruence ρ on a subalgebra of $\mathcal{A}$ and for any congruence θ on $\mathcal{A}$,

$$(\rho \cap \theta)_{\mathcal{A}} = \rho_{\mathcal{A}} \cap \theta.$$

Equivalently, $\mathcal{A}$ has the wCIP if and only if for any $\rho, \theta \in Cw\mathcal{A}$,

$$\Delta \leqslant \theta \text{ implies } \Delta \vee (\rho \wedge \theta) = (\Delta \vee \rho) \wedge \theta.$$

A special case of the CIP is the **weak** congruence intersection property (wCIP).

An algebra $\mathcal{A}$ satisfies the wCIP if for any congruence ρ on a subalgebra of $\mathcal{A}$ and for any congruence θ on $\mathcal{A}$,

$$(\rho \cap \theta)_{\mathcal{A}} = \rho_{\mathcal{A}} \cap \theta.$$

Equivalently, $\mathcal{A}$ has the wCIP if and only if for any $\rho, \theta \in Cw\,\mathcal{A}$,

$$\Delta \leqslant \theta \text{ implies } \Delta \vee (\rho \wedge \theta) = (\Delta \vee \rho) \wedge \theta.$$

That is, $\mathcal{A}$ *has the wCIP if and only if* Δ *is a modular element in the lattice* $\mathrm{Con_w}(\mathcal{A})$.

Varieties satisfying the CIP

Varieties satisfying the CIP

As it is known, an algebra $\mathcal{A}$ is **Abelian** if it satisfies the **term condition** (TC):

Varieties satisfying the CIP

As it is known, an algebra $\mathcal{A}$ is **Abelian** if it satisfies the **term condition** (TC): For each term $t(x, \overline{y})$ in the language of $\mathcal{A}$ and for all $a, b, \overline{c}$ and $\overline{d}$ in A (where $\overline{x}$ stands for the n-tuple $x_1, \ldots, x_n$) if $t(a, \overline{c}) = t(a, \overline{d})$, then $t(b, \overline{c}) = t(b, \overline{d})$.

Varieties satisfying the CIP

As it is known, an algebra $\mathcal{A}$ is **Abelian** if it satisfies the **term condition** (TC): For each term $t(x, \overline{y})$ in the language of $\mathcal{A}$ and for all $a, b, \overline{c}$ and $\overline{d}$ in A (where $\overline{x}$ stands for the n-tuple $x_1, \ldots, x_n$) if $t(a, \overline{c}) = t(a, \overline{d})$, then $t(b, \overline{c}) = t(b, \overline{d})$.

Theorem

If $\mathcal{A}^2$ satisfies the weak CIP, then the algebra $\mathcal{A}$ is Abelian.

Varieties satisfying the CIP

As it is known, an algebra $\mathcal{A}$ is **Abelian** if it satisfies the **term condition** (TC): For each term $t(x, \overline{y})$ in the language of $\mathcal{A}$ and for all $a, b, \overline{c}$ and $\overline{d}$ in A (where $\overline{x}$ stands for the n-tuple $x_1, \ldots, x_n$) if $t(a, \overline{c}) = t(a, \overline{d})$, then $t(b, \overline{c}) = t(b, \overline{d})$.

Theorem

If $\mathcal{A}^2$ satisfies the weak CIP, then the algebra $\mathcal{A}$ is Abelian.

Corollary

Every CIP variety is Abelian.

Varieties satisfying the CIP

As it is known, an algebra $\mathcal{A}$ is **Abelian** if it satisfies the **term condition** (TC): For each term $t(x, \overline{y})$ in the language of $\mathcal{A}$ and for all $a, b, \overline{c}$ and $\overline{d}$ in A (where $\overline{x}$ stands for the n-tuple $x_1, \ldots, x_n$) if $t(a, \overline{c}) = t(a, \overline{d})$, then $t(b, \overline{c}) = t(b, \overline{d})$.

Theorem

If $\mathcal{A}^2$ satisfies the weak CIP, then the algebra $\mathcal{A}$ is Abelian.

Corollary

Every CIP variety is Abelian.

Since *a locally finite variety is Abelian if and only if it is Hamiltonian* (Kiss,Valeriote, 1993), the following holds:

Varieties satisfying the CIP

As it is known, an algebra $\mathcal{A}$ is **Abelian** if it satisfies the **term condition** (TC): For each term $t(x, \overline{y})$ in the language of $\mathcal{A}$ and for all $a, b, \overline{c}$ and $\overline{d}$ in A (where $\overline{x}$ stands for the n-tuple $x_1, \ldots, x_n$) if $t(a, \overline{c}) = t(a, \overline{d})$, then $t(b, \overline{c}) = t(b, \overline{d})$.

Theorem

If $\mathcal{A}^2$ satisfies the weak CIP, then the algebra $\mathcal{A}$ is Abelian.

Corollary

Every CIP variety is Abelian.

Since *a locally finite variety is Abelian if and only if it is Hamiltonian* (Kiss,Valeriote, 1993), the following holds:

Every locally finite CIP variety is Hamiltonian.

Theorem

A CM Abelian variety has the CIP if and only if it has a constant term operation.

Theorem

A CM Abelian variety has the CIP if and only if it has a constant term operation.

Theorem

A CM variety is Abelian if and only if it has the wCIP.

Theorem

A CM Abelian variety has the CIP if and only if it has a constant term operation.

Theorem

A CM variety is Abelian if and only if it has the wCIP.

The above equivalences are not generally satisfied for single algebras in CM varieties. As an example we mention *the eight-element quaternion group, which satisfies the CIP, but fails to be Abelian*.

Theorem

A CM Abelian variety has the CIP if and only if it has a constant term operation.

Theorem

A CM variety is Abelian if and only if it has the wCIP.

The above equivalences are not generally satisfied for single algebras in CM varieties. As an example we mention *the eight-element quaternion group, which satisfies the CIP, but fails to be Abelian*.

Theorem

If a variety $\mathcal{V}$ is CM and SM, then it is a CIP variety.

Theorem

A CM Abelian variety has the CIP if and only if it has a constant term operation.

Theorem

A CM variety is Abelian if and only if it has the wCIP.

The above equivalences are not generally satisfied for single algebras in CM varieties. As an example we mention *the eight-element quaternion group, which satisfies the CIP, but fails to be Abelian*.

Theorem

If a variety $\mathcal{V}$ is CM and SM, then it is a CIP variety.

Problem

Which (possibly locally finite) Abelian (or Hamiltonian) varieties possess the CIP?

Congruence Extension Property, CEP

Congruence Extension Property, CEP

Recall that an algebra $\mathcal{A}$ has the **Congruence Extension Property**, the CEP, if for any congruence ρ on a subalgebra $\mathcal{B}$ of $\mathcal{A}$, there is a congruence θ on $\mathcal{A}$, such that $\rho = B^2 \cap \theta$.

Congruence Extension Property, CEP

Recall that an algebra $\mathcal{A}$ has the **Congruence Extension Property**, the CEP, if for any congruence ρ on a subalgebra $\mathcal{B}$ of $\mathcal{A}$, there is a congruence θ on $\mathcal{A}$, such that $\rho = B^2 \cap \theta$.

Theorem

The following are equivalent for an algebra $\mathcal{A}$ *:*

(*i*) $\mathcal{A}$ *has the CEP;*

(*ii*) *in* $\mathrm{Con_w}(\mathcal{A})$, *for* $\rho, \theta \in \mathrm{Con}\,\mathcal{B}$, $\mathcal{B} \in \mathrm{Sub}\,\mathcal{A}$,
$\rho \vee \Delta = \theta \vee \Delta$ *implies* $\rho = \theta$;

(*iii*) *for* $\rho, \theta \in \mathrm{Con_w}(\mathcal{A})$,
$\rho \leqslant \theta$ *implies* $\rho \vee (\Delta \wedge \theta) = (\rho \vee \Delta) \wedge \theta$;

(*iv*) *for* $\rho \in \mathrm{Con_w}(\mathcal{A})$, $\mathcal{B} \in \mathrm{Sub}\,\mathcal{A}$,
$\rho \leqslant B^2$ *implies* $\rho \vee (\Delta \wedge B^2) = (\rho \vee \Delta) \wedge B^2$;

(*v*) *for* $\rho, \theta \in \mathrm{Con_w}(\mathcal{A})$,
$\rho \vee (\Delta \wedge \theta) = (\rho \vee \Delta) \wedge (\rho \vee \theta)$.

Corollary

An algebra $\mathcal{A}$ has the CIP and the CEP if and only if Δ is a neutral element in the lattice $\mathrm{Con_w}(\mathcal{A})$.

Corollary

An algebra $\mathcal{A}$ has the CIP and the CEP if and only if Δ is a neutral element in the lattice $\mathrm{Con_w}(\mathcal{A})$.

Hence, *$\mathcal{A}$ has both, the CIP and the CEP, if and only if the mapping $p_\Delta : \rho \mapsto (B_\rho, \rho \vee \Delta)$, where $B_\rho = \{x \in A \mid x\,\rho\,x\}$, is an embedding of the lattice $\mathrm{Con_w}(\mathcal{A})$ into the direct product* $\mathrm{Sub}\,\mathcal{A} \times \mathrm{Con}\,\mathcal{A}$.

Corollary

An algebra $\mathcal{A}$ has the CIP and the CEP if and only if Δ is a neutral element in the lattice $\mathrm{Con_w}(\mathcal{A})$.

Hence, *$\mathcal{A}$ has both, the CIP and the CEP, if and only if the mapping $p_\Delta : \rho \mapsto (B_\rho, \rho \vee \Delta)$, where $B_\rho = \{x \in A \mid x\,\rho\,x\}$, is an embedding of the lattice* $\mathrm{Con_w}(\mathcal{A})$ *into the direct product* $\mathrm{Sub}\,\mathcal{A} \times \mathrm{Con}\,\mathcal{A}$.
If Δ has a complement, i.e., if it belongs to the center of $\mathrm{Con_w}(\mathcal{A})$, *then p_Δ is an isomorphism.*

Corollary

An algebra $\mathcal{A}$ has the CIP and the CEP if and only if Δ is a neutral element in the lattice $\mathrm{Con_w}(\mathcal{A})$.

Hence, *$\mathcal{A}$ has both, the CIP and the CEP, if and only if the mapping $p_\Delta : \rho \mapsto (B_\rho, \rho \vee \Delta)$, where $B_\rho = \{x \in A \mid x\,\rho\,x\}$, is an embedding of the lattice $\mathrm{Con_w}(\mathcal{A})$ into the direct product* $\mathrm{Sub}\,\mathcal{A} \times \mathrm{Con}\,\mathcal{A}$.

If Δ has a complement, i.e., if it belongs to the center of $\mathrm{Con_w}(\mathcal{A})$, *then p_Δ is an isomorphism.*

Theorem

If an algebra $\mathcal{A}$ has the CIP and the CEP, then any lattice identity holds on $\mathrm{Con_w}(\mathcal{A})$ if and only if it holds on $\mathrm{Sub}\,\mathcal{A}$ and on $\mathrm{Con}\,\mathcal{A}$.

$*$CIP

$*$CIP

An algebra $\mathcal{A}$ is said to posses the $*$**CIP** if for any family $\{\rho_i \mid i \in I\}$ of congruences on subalgebras of $\mathcal{A}$ ($\rho_i \in \mathcal{A}_i$),

$$\bigcap(\rho_i \mid i \in I)_{\mathcal{A}} = \bigcap((\rho_i)_{\mathcal{A}} \mid i \in I).$$

$*$CIP

An algebra $\mathcal{A}$ is said to posses the $*$**CIP** if for any family $\{\rho_i \mid i \in I\}$ of congruences on subalgebras of $\mathcal{A}$ ($\rho_i \in \mathcal{A}_i$),

$$\bigcap(\rho_i \mid i \in I)_{\mathcal{A}} = \bigcap((\rho_i)_{\mathcal{A}} \mid i \in I).$$

Obviously, *$\mathcal{A}$ has the $*$CIP if and only if Δ is an infinitely distributive element in the lattice* $\mathrm{Con_w}(\mathcal{A})$, i.e., if

$$\Delta \vee \bigwedge_{i \in I} \rho_i = \bigwedge_{i \in I} (\Delta \vee \rho_i).$$

Lattice identities in $\mathrm{Con_w}(\mathcal{A})$

Lattice identities in $\mathrm{Con_w}(\mathcal{A})$

Proposition

If an algebra $\mathcal{A}$ has the CIP and the CEP, and $\mathrm{Sub}\,\mathcal{A}$ and $\mathrm{Con}\,\mathcal{A}$ are modular (distributive) lattices, then also its lattice of weak congruences is modular (distributive).

Lattice identities in $\mathrm{Con}_w(\mathcal{A})$

Proposition

If an algebra $\mathcal{A}$ has the CIP and the CEP, and Sub $\mathcal{A}$ and Con $\mathcal{A}$ are modular (distributive) lattices, then also its lattice of weak congruences is modular (distributive).

For the converse, observe that in a modular lattice every codistributive element is neutral.

Lattice identities in $\mathrm{Con}_w(\mathcal{A})$

Proposition

If an algebra $\mathcal{A}$ has the CIP and the CEP, and $\mathrm{Sub}\,\mathcal{A}$ and $\mathrm{Con}\,\mathcal{A}$ are modular (distributive) lattices, then also its lattice of weak congruences is modular (distributive).

For the converse, observe that in a modular lattice every codistributive element is neutral.

Theorem

An algebra $\mathcal{A}$ has modular (distributive) lattice of weak congruences if and only if $\mathrm{Sub}\,\mathcal{A}$ and $\mathrm{Con}\,\mathcal{A}$ are modular (distributive) lattices and $\mathcal{A}$ has the CIP and the CEP.

Theorem

The lattice of weak congruences of an algebra $\mathcal{A}$ is relatively complemented if and only if all of the following conditions are satisfied:

- $\mathcal{A}$ has at least one nullary operation,

- no nontrivial congruence on $\mathcal{A}$ has a block which is a subalgebra of $\mathcal{A}$,

- $\mathcal{A}$ satisfies the CEP and the CIP, and

- both $\operatorname{Sub}\mathcal{A}$ and $\operatorname{Con}\mathcal{A}$ are relatively complemented lattices.

Theorem

Let $\mathcal{A}$ be an algebra which has the CIP. Then the weak congruence lattice of $\mathcal{A}$ is complemented if and only if the following conditions hold:
- $\mathcal{A}$ has at least one nullary operation;
- no congruence on $\mathcal{A}$ has a block which is a proper subalgebra of $\mathcal{A}$;
- $\mathrm{Sub}\,\mathcal{A}$ and $\mathrm{Con}\,\mathcal{A}$ are complemented lattices.

Theorem

Let $\mathcal{A}$ be an algebra which has the CIP. Then the weak congruence lattice of $\mathcal{A}$ is complemented if and only if the following conditions hold:
- $\mathcal{A}$ has at least one nullary operation;
- no congruence on $\mathcal{A}$ has a block which is a proper subalgebra of $\mathcal{A}$;
- Sub $\mathcal{A}$ *and* Con $\mathcal{A}$ *are complemented lattices.*

Corollary

The weak congruence lattice of an algebra $\mathcal{A}$ is Boolean if and only if $\mathcal{A}$ satisfies conditions:
(i) *for every subalgebra $\mathcal{B}$,* Con $\mathcal{B}$ *is isomorphic with* Con $\mathcal{A}$*, under* $\rho \mapsto \rho_A$ *and*
(ii) Sub $\mathcal{A}$ *and* Con $\mathcal{A}$ *are Boolean lattices.*

Weak congruences on groups and rings

Weak congruences on groups and rings

For every group $\mathcal{G}$ there is a 1-1 correspondence between weak congruences and ordered pairs (H, K) of subgroups of $\mathcal{G}$, such that $K \triangleleft H$.

Theorem (Czédli, Šešelja, Tepavčević, 2009)

For any finite group G the following five conditions are equivalent.
(i) G is a Dedekind group;
(ii) G has the CIP;
(iii) Δ *is a join-semidistributive element in* $\mathrm{Con_w}(G)$*;*
(iv) for every normal subgroup N of G,

$$C_N := \{K \in \mathrm{Sub}(G) : \exists H \in \mathrm{Sub}_N(K) \ \textit{with} \ (H)_G = N\}$$

is a sublattice of $\mathrm{Sub}(G)$*;*
(v) for every normal subgroup N of G, C_N *is closed with respect to intersection.*

Theorem (Czédli, Erné, Šešelja, Tepavčević, 2009)

The following statements on a group G are equivalent:

(1) *G is a Dedekind group.*

(2) $\mathrm{Con_w}(G)$ *is modular.*

(3) Δ *is a standard (equivalently, a neutral) element of* $\mathrm{Con_w}(G)$.

(4) *G has the* CIP *and the* CEP.

Theorem (Czédli, Erné, Šešelja, Tepavčević, 2009)

The following statements on a group G are equivalent:

(1) *G is a Dedekind group.*

(2) $\mathrm{Con}_\mathrm{w}(G)$ *is modular.*

(3) Δ *is a standard (equivalently, a neutral) element of* $\mathrm{Con}_\mathrm{w}(G)$.

(4) *G has the* CIP *and the* CEP.

Recall that Ore's Theorem says that the locally cyclic groups are exactly those with a distributive subgroup lattice.

Theorem (Czédli, Erné, Šešelja, Tepavčević, 2009)

The following statements on a group G are equivalent:

(1) *G is a Dedekind group.*

(2) $\mathrm{Con_w}(G)$ *is modular.*

(3) Δ *is a standard (equivalently, a neutral) element of* $\mathrm{Con_w}(G)$.

(4) *G has the* CIP *and the* CEP.

Recall that Ore's Theorem says that the locally cyclic groups are exactly those with a distributive subgroup lattice.

Corollary

A group is locally cyclic if and only if its weak congruence lattice is distributive.

We call a ring **Hamiltonian** if each subring is an ideal.

We call a ring **Hamiltonian** if each subring is an ideal.

Theorem (Czédli, Erné, Šešelja, Tepavčević, 2009)

A ring is Hamiltonian if and only if it is generated by its Hamiltonian subrings and has a modular weak congruence lattice (or Δ is a neutral element of it).

We call a ring **Hamiltonian** if each subring is an ideal.

Theorem (Czédli, Erné, Šešelja, Tepavčević, 2009)

A ring is Hamiltonian if and only if it is generated by its Hamiltonian subrings and has a modular weak congruence lattice (or Δ is a neutral element of it).

Example

We call a ring **Hamiltonian** if each subring is an ideal.

Theorem (Czédli, Erné, Šešelja, Tepavčević, 2009)

A ring is Hamiltonian if and only if it is generated by its Hamiltonian subrings and has a modular weak congruence lattice (or Δ is a neutral element of it).

Example

In the ring $\mathbb{Z}$ of all integers, the subrings coincide with the additive subgroups $n\mathbb{Z}$ and with the ideals. Thus $\mathbb{Z}$ is Hamiltonian. The weak congruence lattice $\mathrm{Con}_w(\mathbb{Z})$ is distributive, being isomorphic to

$$D_{\geq} = \{(x, y) \in D^2 \mid x \geq y\},$$

where D is the lattice of all natural numbers (including 0), ordered by the dual of the divisibility relation.

Proposition

Every module satisfies the CEP and the CIP. In addition, the lattice of weak congruences of a module is modular.

Proposition

Every module satisfies the CEP and the CIP. In addition, the lattice of weak congruences of a module is modular.

Theorem (Chajda, Šešelja, Tepavčević, 1995)

The variety of modules is weak congruence modular.

Proposition

Every module satisfies the CEP and the CIP. In addition, the lattice of weak congruences of a module is modular.

Theorem (Chajda, Šešelja, Tepavčević, 1995)

The variety of modules is weak congruence modular.

Theorem (Chajda, Šešelja, Tepavčević, 1995)

A variety $\mathcal{V}$ which has a nullary operation in the similarity type is weak congruence modular if and only if $\mathcal{V}$ is polynomially equivalent to the variety of modules over a ring with unit.

The CIP for groups

The CIP for groups

If H is a subgroup of a group $\mathcal{G}$, then let $\overline{H}$ be the normal closure of H, i.e., the smallest normal subgroup of $\mathcal{G}$ containing H.

The CIP for groups

If H is a subgroup of a group $\mathcal{G}$, then let $\overline{H}$ be the normal closure of H, i.e., the smallest normal subgroup of $\mathcal{G}$ containing H.

Then $\mathcal{G}$ has the CIP if and only if for every pair of subgroups H, K,

$$\overline{H \cap K} = \overline{H} \cap \overline{K}.$$

The CIP for groups

If H is a subgroup of a group $\mathcal{G}$, then let $\overline{H}$ be the normal closure of H, i.e., the smallest normal subgroup of $\mathcal{G}$ containing H.

Then $\mathcal{G}$ has the CIP if and only if for every pair of subgroups H, K,

$$\overline{H \cap K} = \overline{H} \cap \overline{K}.$$

Analogously, $\mathcal{G}$ satisfies the $*$CIP if and only if

$$\overline{\bigcap_{i \in I} H_i} = \bigcap_{i \in I} \overline{H_i},$$

for every family $\{H_i \mid i \in I\}$ of subgroups.

The CIP for groups

If H is a subgroup of a group $\mathcal{G}$, then let $\overline{H}$ be the normal closure of H, i.e., the smallest normal subgroup of $\mathcal{G}$ containing H.
Then $\mathcal{G}$ has the CIP if and only if for every pair of subgroups H, K,

$$\overline{H \cap K} = \overline{H} \cap \overline{K}.$$

Analogously, $\mathcal{G}$ satisfies the $*$CIP if and only if

$$\overline{\bigcap_{i \in I} H_i} = \bigcap_{i \in I} \overline{H_i},$$

for every family $\{H_i \mid i \in I\}$ of subgroups.

Theorem

A finite group $\mathcal{G}$ is a Dedekind group if and only if it satisfies the CIP.

The CIP for groups

If H is a subgroup of a group $\mathcal{G}$, then let $\overline{H}$ be the normal closure of H, i.e., the smallest normal subgroup of $\mathcal{G}$ containing H.

Then $\mathcal{G}$ has the CIP if and only if for every pair of subgroups H, K,

$$\overline{H \cap K} = \overline{H} \cap \overline{K}.$$

Analogously, $\mathcal{G}$ satisfies the $*$CIP if and only if

$$\overline{\bigcap_{i \in I} H_i} = \bigcap_{i \in I} \overline{H_i},$$

for every family $\{H_i \mid i \in I\}$ of subgroups.

Theorem

A finite group $\mathcal{G}$ is a Dedekind group if and only if it satisfies the CIP.

Theorem

A group $\mathcal{G}$ is a Dedekind group if and only if it satisfies the $*$CIP.

Problem

Is it true that a group is a Dedekind one if and only if it satisfies the CIP?

Problem

Is it true that a group is a Dedekind one if and only if it satisfies the CIP?

V.N. Obraztsov (1998) proved that *there exists a group $\mathcal{G}$ such that*

- $\mathcal{G}$ is torsion-free;

- in $\mathcal{G}$ every two non trivial cyclic subgroups have a non-trivial intersection;

- $\mathcal{G}$ is simple.

Problem

Is it true that a group is a Dedekind one if and only if it satisfies the CIP?

V.N. Obraztsov (1998) proved that *there exists a group $\mathcal{G}$ such that*

- $\mathcal{G}$ is torsion-free;

- in $\mathcal{G}$ every two non trivial cyclic subgroups have a non-trivial intersection;

- $\mathcal{G}$ is simple.

Obviously such a group has the CIP but it is not a Dedekind one.

Representation of lattices by weak congruences

Representation of lattices by weak congruences

Bacic representation problem

Represent an algebraic lattice by a weak congruence lattice of an algebra.

Representation of lattices by weak congruences

Bacic representation problem

Represent an algebraic lattice by a weak congruence lattice of an algebra.

Easily solved by Grätzer-Schmidt theorem:
Let $\mathcal{B} = (A, F)$ be an algebra such that $\operatorname{Con} \mathcal{B}$ is isomorphic with L. Then the required algebra $\mathcal{A}$ can be obtained by adding to F all the elements from A as nullary operations: $\mathcal{A} = (A, F \cup \{A\})$.
Obviously, $\operatorname{Con_w}(\mathcal{A}) \cong \operatorname{Con} \mathcal{B} \cong L$.

Representation of lattices by weak congruences

Bacic representation problem

Represent an algebraic lattice by a weak congruence lattice of an algebra.

Easily solved by Grätzer-Schmidt theorem:
Let $\mathcal{B} = (A, F)$ be an algebra such that $\mathrm{Con}\,\mathcal{B}$ is isomorphic with L. Then the required algebra $\mathcal{A}$ can be obtained by adding to F all the elements from A as nullary operations: $\mathcal{A} = (A, F \cup \{A\})$.
Obviously, $\mathrm{Con_w}(\mathcal{A}) \cong \mathrm{Con}\,\mathcal{B} \cong L$.

The above construction by which the diagonal relation of the algebra corresponds to the bottom of the lattice is called the **trivial representation**.

Weak congruence lattice representation problem

Let L be an algebraic lattice and $a \in L$. Find an algebra such that its weak congruence lattice is isomorphic with L, the diagonal relation being the image of a under the isomorphism.

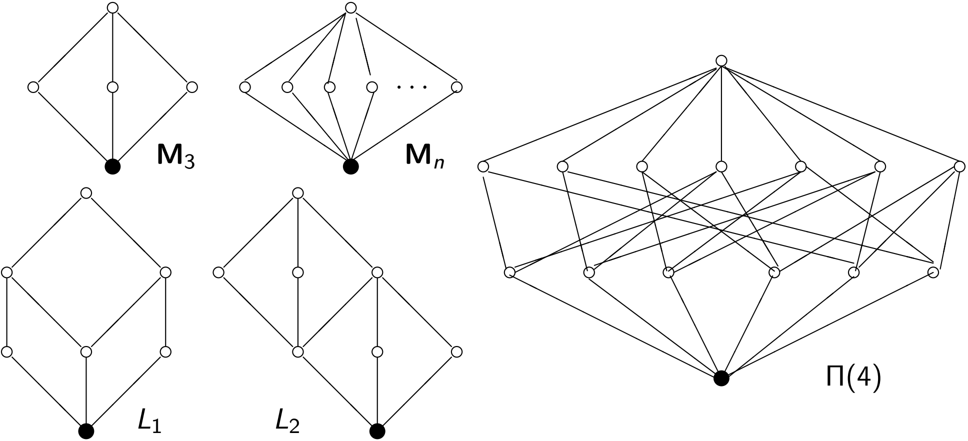
Weak congruence lattice representation problem

Let L be an algebraic lattice and $a \in L$. Find an algebra such that its weak congruence lattice is isomorphic with L, the diagonal relation being the image of a under the isomorphism.

A representation by which the diagonal relation corresponds to an element different from the bottom of the lattice is said to be **non-trivial**.

Examples: lattices without non-trivial representations

Examples: lattices without non-trivial representations



Δ-suitable elements of a lattice

Δ-suitable elements of a lattice

Let L be an algebraic lattice. An element $a \in L$ is said to be **Δ-suitable** if there is an algebra $\mathcal{A}$ such that the weak congruence lattice $\mathrm{Con_w}(\mathcal{A})$ is isomorphic to L, and Δ corresponds to a under the isomorphism.

Δ-suitable elements of a lattice

Let L be an algebraic lattice. An element $a \in L$ is said to be **Δ-suitable** if there is an algebra $\mathcal{A}$ such that the weak congruence lattice $\mathrm{Con}_w(\mathcal{A})$ is isomorphic to L, and Δ corresponds to a under the isomorphism.

Proposition

Every Δ-suitable element of a lattice is co-distributive.

Proposition

A Δ-suitable element $a \in L$ satisfies the following:

Proposition

A Δ-suitable element $a \in L$ satisfies the following:

- *if $x \wedge y \neq \mathbf{0}$ then $\overline{x \vee y} = \overline{x} \vee \overline{y}$;*

Proposition

A Δ-suitable element $a \in L$ satisfies the following:

- *if $x \wedge y \neq \mathbf{0}$ then $\overline{x \vee y} = \overline{x} \vee \overline{y}$;*
- *if $x \neq \mathbf{0}$ and $\overline{x} < y$, then $\overline{y \wedge a} \neq y \wedge a$;*

Proposition

A Δ-suitable element $a \in L$ satisfies the following:

- *if $x \wedge y \neq \mathbf{0}$ then $\overline{x \vee y} = \overline{x} \vee \overline{y}$;*
- *if $x \neq \mathbf{0}$ and $\overline{x} < y$, then $\overline{y \wedge a} \neq y \wedge a$;*
- *if $x \prec a$, then $\bigvee(y \in \uparrow a \mid y \vee \overline{x} < \mathbf{1}) \neq \mathbf{1}$;*

Proposition

A Δ-suitable element $a \in L$ satisfies the following:

- *if* $x \wedge y \neq \mathbf{0}$ *then* $\overline{x \vee y} = \overline{x} \vee \overline{y}$;
- *if* $x \neq \mathbf{0}$ *and* $\overline{x} < y$, *then* $\overline{y \wedge a} \neq y \wedge a$;
- *if* $x \prec a$, *then* $\bigvee (y \in \uparrow a \mid y \vee \overline{x} < \mathbf{1}) \neq \mathbf{1}$;
- *If* $y \in \downarrow a$ *and* $x \prec y$, *then there exists* $z \in [y, \overline{y}]$, *such that*
  - *for all* $t \in [x, \overline{x}]$, *the set* $\{c \in \mathrm{Ext}(t) \mid c \leqslant z\}$ *is either empty or has the top element, and*
  - *for all* $t \in [x, \overline{x}]$, *the set* $\{c \in \mathrm{Ext}(t) \mid c \nleqslant z\}$ *is an antichain (possibly empty), where*

$$\mathrm{Ext}(t) := \{w \in [y, \overline{y}] \mid w \cap \overline{x} = t\}.$$

Proposition

If a is a Δ-suitable element of the lattice L, then the following hold:

Proposition

If a is a Δ-suitable element of the lattice L, then the following hold:

- $x \wedge a < y \wedge a$ *implies* $\overline{x} \vee a < \overline{y} \vee a$ *for all* $x, y \in L$ *if and only if every algebra representing L is Hamiltonian;*

Proposition

If a is a Δ-suitable element of the lattice L, then the following hold:

- $x \wedge a < y \wedge a$ *implies* $\overline{x} \vee a < \overline{y} \vee a$ *for all* $x, y \in L$ *if and only if every algebra representing* L *is Hamiltonian;*
- a *is a cancellable element in* L *if and only if every algebra representing* L *has the CEP;*

Proposition

If a is a Δ-suitable element of the lattice L, then the following hold:

- $x \wedge a < y \wedge a$ *implies* $\overline{x} \vee a < \overline{y} \vee a$ *for all* $x, y \in L$ *if and only if every algebra representing* L *is Hamiltonian;*
- a *is a cancellable element in* L *if and only if every algebra representing* L *has the CEP;*
- a *is a distributive element in* L *if and only if every algebra representing* L *has the CIP;*

Proposition

If a is a Δ-suitable element of the lattice L, then the following hold:

- $x \wedge a < y \wedge a$ *implies* $\overline{x} \vee a < \overline{y} \vee a$ *for all* $x, y \in L$ *if and only if every algebra representing* L *is Hamiltonian;*
- a *is a cancellable element in* L *if and only if every algebra representing* L *has the CEP;*
- a *is a distributive element in* L *if and only if every algebra representing* L *has the CIP;*
- $\overline{x} \vee a = \mathbf{1}$ *for every* $x \in L$ *if and only if no congruence on an algebra representing* L *has a block which is a proper subalgebra;*

- *$x \prec a$ implies $\overline{x} \vee a < 1$ for every $x \in L$ if and only if every algebra representing L is quasi-Hamiltonian;*

- *$x \prec a$ implies $\overline{x} \vee a < 1$ for every $x \in L$ if and only if every algebra representing L is quasi-Hamiltonian;*
- *a has a complement in L if and only if every algebra representing L has at least one nullary operation and has no congruence whose block is a proper subalgebra.*

Theorem

If a is a Δ-suitable element belonging to the center of a lattice L, then every algebra representing L satisfies the following:

Theorem

If a is a Δ-suitable element belonging to the center of a lattice L, then every algebra representing L satisfies the following:

- $\mathcal{A}$ *has at least one nullary operation;*

Theorem

If a is a Δ-suitable element belonging to the center of a lattice L, then every algebra representing L satisfies the following:

- *$\mathcal{A}$ has at least one nullary operation;*
- *$\mathcal{A}$ has the CEP and the CIP;*

Theorem

If a is a Δ-suitable element belonging to the center of a lattice L, then every algebra representing L satisfies the following:

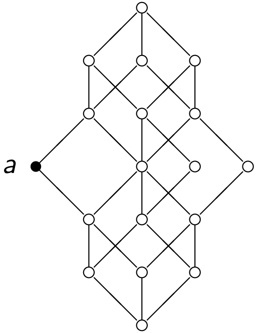
- *$\mathcal{A}$ has at least one nullary operation;*
- *$\mathcal{A}$ has the CEP and the CIP;*
- *for every subalgebra $\mathcal{B}$ of $\mathcal{A}$, $\operatorname{Con}\mathcal{B}$ is isomorphic with $\operatorname{Con}\mathcal{A}$;*

Theorem

If a is a Δ-suitable element belonging to the center of a lattice L, then every algebra representing L satisfies the following:

- *$\mathcal{A}$ has at least one nullary operation;*
- *$\mathcal{A}$ has the CEP and the CIP;*
- *for every subalgebra $\mathcal{B}$ of $\mathcal{A}$, $\mathrm{Con}\,\mathcal{B}$ is isomorphic with $\mathrm{Con}\,\mathcal{A}$;*
- *$\mathcal{A}$ is not Hamiltonian, moreover no congruence on $\mathcal{A}$ has a block which is a subalgebra of $\mathcal{A}$.*

Example



In the free distributive lattice with three generators, the generating elements (and, trivially, the bottom) are the only ones which are Δ-suitable.

In the free distributive lattice with three generators, the generating elements (and, trivially, the bottom) are the only ones which are Δ*-suitable.*
Hence, there is one possible non-trivial representation of this lattice.

In the free distributive lattice with three generators, the generating elements (and, trivially, the bottom) are the only ones which are Δ*-suitable.*
Hence, there is one possible non-trivial representation of this lattice.
Every algebra representing this lattice:

In the free distributive lattice with three generators, the generating elements (and, trivially, the bottom) are the only ones which are Δ-suitable.
Hence, there is one possible non-trivial representation of this lattice.
Every algebra representing this lattice:

- *is non-Hamiltonian;*

In the free distributive lattice with three generators, the generating elements (and, trivially, the bottom) are the only ones which are Δ-suitable.
Hence, there is one possible non-trivial representation of this lattice.
Every algebra representing this lattice:

- *is non-Hamiltonian;*
- *has nullary operations;*

In the free distributive lattice with three generators, the generating elements (and, trivially, the bottom) are the only ones which are Δ-suitable.
Hence, there is one possible non-trivial representation of this lattice.
Every algebra representing this lattice:

- *is non-Hamiltonian;*
- *has nullary operations;*
- *satisfy the CEP and the CIP;*

In the free distributive lattice with three generators, the generating elements (and, trivially, the bottom) are the only ones which are Δ-suitable.
Hence, there is one possible non-trivial representation of this lattice.
Every algebra representing this lattice:

- *is non-Hamiltonian;*
- *has nullary operations;*
- *satisfy the CEP and the CIP;*
- *all its proper subalgebras are Hamiltonian.*

In a complete and atomic Boolean algebra $\mathcal{B}$ all elements except the top are Δ-suitable.

In a complete and atomic Boolean algebra $\mathcal{B}$ all elements except the top are Δ-suitable.
Hence, for every $a \in B$, $a \neq \mathbf{1}$, there could be a representation of $\mathcal{B}$ by weak congruences.

In a complete and atomic Boolean algebra $\mathcal{B}$ all elements except the top are Δ-suitable.
Hence, for every $a \in B$, $a \neq \mathbf{1}$, there could be a representation of $\mathcal{B}$ by weak congruences.
The representation is not trivial if $a \neq \mathbf{0}$. In this case, every algebra $\mathcal{A}$ representing $\mathcal{B}$ has the following properties:

In a complete and atomic Boolean algebra $\mathcal{B}$ all elements except the top are Δ-suitable.
Hence, for every $a \in B$, $a \neq \mathbf{1}$, there could be a representation of $\mathcal{B}$ by weak congruences.
The representation is not trivial if $a \neq \mathbf{0}$. In this case, every algebra $\mathcal{A}$ representing $\mathcal{B}$ has the following properties:

- *$\mathcal{A}$ has at least one nullary operation in its similarity type;*

In a complete and atomic Boolean algebra $\mathcal{B}$ all elements except the top are Δ-suitable.
Hence, for every $a \in B$, $a \neq \mathbf{1}$, there could be a representation of $\mathcal{B}$ by weak congruences.
The representation is not trivial if $a \neq \mathbf{0}$. In this case, every algebra $\mathcal{A}$ representing $\mathcal{B}$ has the following properties:

- *$\mathcal{A}$ has at least one nullary operation in its similarity type;*
- *$\mathcal{A}$ satisfies the CEP and the CIP;*

In a complete and atomic Boolean algebra $\mathcal{B}$ all elements except the top are Δ-suitable.
Hence, for every $a \in B$, $a \neq \mathbf{1}$, there could be a representation of $\mathcal{B}$ by weak congruences.
The representation is not trivial if $a \neq \mathbf{0}$. In this case, every algebra $\mathcal{A}$ representing $\mathcal{B}$ has the following properties:

- *$\mathcal{A}$ has at least one nullary operation in its similarity type;*
- *$\mathcal{A}$ satisfies the CEP and the CIP;*
- *$\mathcal{A}$ is not Hamiltonian, neither quasi-Hamiltonian;*

In a complete and atomic Boolean algebra $\mathcal{B}$ all elements except the top are Δ-suitable.
Hence, for every $a \in B$, $a \neq \mathbf{1}$, there could be a representation of $\mathcal{B}$ by weak congruences.
The representation is not trivial if $a \neq \mathbf{0}$. In this case, every algebra $\mathcal{A}$ representing $\mathcal{B}$ has the following properties:

- *$\mathcal{A}$ has at least one nullary operation in its similarity type;*
- *$\mathcal{A}$ satisfies the CEP and the CIP;*
- *$\mathcal{A}$ is not Hamiltonian, neither quasi-Hamiltonian;*
- *no congruence on $\mathcal{A}$ has a block which is a subalgebra of $\mathcal{A}$;*

In a complete and atomic Boolean algebra $\mathcal{B}$ all elements except the top are Δ-suitable.
Hence, for every $a \in B$, $a \neq \mathbf{1}$, there could be a representation of $\mathcal{B}$ by weak congruences.
The representation is not trivial if $a \neq \mathbf{0}$. In this case, every algebra $\mathcal{A}$ representing $\mathcal{B}$ has the following properties:

- *$\mathcal{A}$ has at least one nullary operation in its similarity type;*
- *$\mathcal{A}$ satisfies the CEP and the CIP;*
- *$\mathcal{A}$ is not Hamiltonian, neither quasi-Hamiltonian;*
- *no congruence on $\mathcal{A}$ has a block which is a subalgebra of $\mathcal{A}$;*
- *all congruence lattices of subalgebras of $\mathcal{A}$ are isomorphic with $\mathrm{Con}\,\mathcal{A}$.*

Theorem

The weak congruence lattice of an algebra $\mathcal{A}$ is atomistic if and only if all the following conditions are fulfilled:

1. *The subalgebra lattice of $\mathcal{A}$ is atomistic;*
2. *$\mathcal{A}$ has the smallest nontrivial subalgebra $\mathcal{B}_m$ whose congruence lattice is atomistic;*
3. *Every congruence on every subalgebra is an extension of a congruence on the smallest subalgebra.*

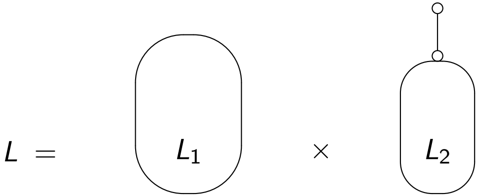
Particular solution

Particular solution

A nontrivial representation problem is solved in a case $L = L_1 \times L_2$ where L_1 and L_2 are algebraic lattices, L_2 having a single co-atom.

Particular solution

A nontrivial representation problem is solved in a case $L = L_1 \times L_2$ where L_1 and L_2 are algebraic lattices, L_2 having a single co-atom.



Theorem (Šešelja, Stepanović, Tepavčević)

Let L be an algebraic lattice and $a \in L$ an element from the center of the lattice, such that $\uparrow a$ has a single co-atom.

Theorem (Šešelja, Stepanović, Tepavčević)

Let L be an algebraic lattice and $a \in L$ an element from the center of the lattice, such that $\uparrow a$ has a single co-atom.
Then, there is an algebra $\mathcal{A}$, whose weak congruence lattice $\mathrm{Con_w}(\mathcal{A})$ is isomorphic with L under a mapping sending Δ to a.

References

References

- Tran Duc Mai, Partitions and congruences in algebras. I Basic properties, II, III Archivum Math. 10 (1974) 111-122.
- G. Vojvodić, B. Šešelja, On the lattice of weak congruence relations, Algebra Univers. 25 (1988) 121-130.
- B. Šešelja, G. Vojvodić, On the complementedness of the lattice of weak congruences, Studia Sci. Mat. Hung. 24 (1989) 289-293.
- M. Ploščica, Graphical compositions and weak congruences, Publ. Inst. Math. Beograd 56(70) 1994 34-40.
- I. Chajda, B. Šešelja, A. Tepavčević, On weak congruence modular varieties, Filomat (Niš) 9:3 (1995) 633-638.
- G. Traustason, CIP-groups, Arch. Math. 65 (1995) 97-102.
- A. Lucchini, O. Puglisi, A note on groups satisfying the congruence intersection property, Proceedings of the Royal Irish Academy 97A n.1 (1997) 61-68.

- N.V. Obraztsov, Simple torsion-free groups in which the intersection of any two non-trivial subgroups is non-trivial. J. Algebra 199 (1998) no. 1 337-343.
- I. Chajda, B. Šešelja, A. Tepavčević, Weak congruences in algebras having restricted similarity types, Discussiones Math. 18 (1998) 27-38.
- I. Chajda, B. Šešelja, A. Tepavčević, Lattices of compatible relations satisfying a set of formulas, Algebra Univers. 40 (1998) 51-58.
- B. Šešelja, A. Tepavčević, Relative complements in the lattice of weak congruences, Publ. Inst. Math. Beograd 67 (81) (2000) 7-13.
- B. Šešelja, A. Tepavčević, A note on CIP varieties, Algebra Univers. 45 (2001) 349-351.
- G. Czédli, A. Walendziak, Subdirect representation and semimodularity of weak congruence lattices, Algebra Univers. 44.3-4 (2000): 371-373.

- A. Walendziak, Subalgebras and homomorphic images of algebras having the CEP and the WCIP, Czech. Math. J., 54 (129) (2004), 155–160
- B. Šešelja, A. Tepavčević, Weak Congruences in Universal Algebra. Institute of Mathematics, Novi Sad, 2001.
- B. Šešelja, A. Tepavčević, A note on atomistic weak congruence lattices. Discrete Mathematics 308, 10 2054–2057 (2008).
- G. Czédli, B. Šešelja, A. Tepavčević, Semidistributive elements in lattices; application to groups and rings, Algebra Univers. 58 (2008) 349-355.
- M. Erné, G. Czédli, B. Šešelja, A. Tepavčević, Characteristic triangles of closure operators with applications in general algebra, Algebra Univers. 62 (2009) 399–418.
- B. Šešelja, V. Stepanović, A. Tepavčević, A note on representation of lattices by weak congruences, Algebra Univers. 68 (2012) 287–291.

Introduction to Ω-structures

Introduction to Ω-structures

In 1977., Fourman and Scott introduced models for intuitionistic predicate logic. These were Ω-*sets*, or *Heyting-valued sets*, Ω being a Heyting algebra.

Introduction to Ω-structures

In 1977., Fourman and Scott introduced models for intuitionistic predicate logic. These were Ω*-sets*, or *Heyting-valued sets*, Ω being a Heyting algebra.

- M.P. Fourman and D.S. Scott, *Sheaves and logic*, Lecture Notes in Mathematics, vol. 753, Springer, Berlin, Heidelberg, New York, 1979, 302–401.

Introduction to Ω-structures

In 1977., Fourman and Scott introduced models for intuitionistic predicate logic. These were Ω*-sets*, or *Heyting-valued sets*, Ω being a Heyting algebra.

- M.P. Fourman and D.S. Scott, *Sheaves and logic*, Lecture Notes in Mathematics, vol. 753, Springer, Berlin, Heidelberg, New York, 1979, 302–401.

  See also

Introduction to Ω-structures

In 1977., Fourman and Scott introduced models for intuitionistic predicate logic. These were Ω*-sets*, or *Heyting-valued sets*, Ω being a Heyting algebra.

- M.P. Fourman and D.S. Scott, *Sheaves and logic*, Lecture Notes in Mathematics, vol. 753, Springer, Berlin, Heidelberg, New York, 1979, 302–401.

  See also

- F. Borceux, R. Cruciani, *Skew Omega-sets coincide with Omega-posets*, Cahiers de topologie et géométrie différentielle catégoriques 39.3 (1998): 205-220, and

Introduction to Ω-structures

In 1977., Fourman and Scott introduced models for intuitionistic predicate logic. These were Ω*-sets*, or *Heyting-valued sets*, Ω being a Heyting algebra.

- M.P. Fourman and D.S. Scott, *Sheaves and logic*, Lecture Notes in Mathematics, vol. 753, Springer, Berlin, Heidelberg, New York, 1979, 302–401.

  See also

- F. Borceux, R. Cruciani, *Skew Omega-sets coincide with Omega-posets*, Cahiers de topologie et géométrie différentielle catégoriques 39.3 (1998): 205-220, and
- F. Borceux, *A hanbook of categorical algebra, volume 3, categories of sheaves*, Cambridge University Press, 1994.

An Ω-set is a nonempty set equipped with an Ω-valued equality.

An Ω-set is a nonempty set equipped with an Ω-valued equality.

The notion has been further applied to non-classical predicate logics e.g.:

An Ω-set is a nonempty set equipped with an Ω-valued equality.

The notion has been further applied to non-classical predicate logics e.g.:

- G.P. Monro, *Quasitopoi, logic and Heyting-valued models*, Journal of pure and applied algebra, (1986) 42(2), 141-164,
- E. Palmgren, S.J. Vickers, *Partial Horn logic and cartesian categories*, Annals of Pure and Applied Logic, (2007) 145(3), 314-353.

Ω-sets and related notions were also applied to foundations of Fuzzy Set Theory, e.g.:

Ω-sets and related notions were also applied to foundations of Fuzzy Set Theory, e.g.:

- U. Höhle, *Fuzzy sets and sheaves. Part I: basic concepts*, Fuzzy Sets and Systems, (2007) 158(11),

Ω-sets and related notions were also applied to foundations of Fuzzy Set Theory, e.g.:

- U. Höhle, *Fuzzy sets and sheaves. Part I: basic concepts*, Fuzzy Sets and Systems, (2007) 158(11),
- S. Gottwald, *Universes of fuzzy sets and axiomatizations of fuzzy set theory, Part II: Category theoretic approaches*, Studia Logica, (2006) 84(1), 23-50. 1143-1174.

Ω-sets and related notions were also applied to foundations of Fuzzy Set Theory, e.g.:

- U. Höhle, *Fuzzy sets and sheaves. Part I: basic concepts*, Fuzzy Sets and Systems, (2007) 158(11),
- S. Gottwald, *Universes of fuzzy sets and axiomatizations of fuzzy set theory, Part II: Category theoretic approaches*, Studia Logica, (2006) 84(1), 23-50. 1143-1174.
- R. Bělohlávek, *Fuzzy Relational Systems: Foundations and Principles*, Kluwer Academic/Plenum Publishers, New York, 2002.

Ω-sets and related notions were also applied to foundations of Fuzzy Set Theory, e.g.:

- U. Höhle, *Fuzzy sets and sheaves. Part I: basic concepts*, Fuzzy Sets and Systems, (2007) 158(11),
- S. Gottwald, *Universes of fuzzy sets and axiomatizations of fuzzy set theory, Part II: Category theoretic approaches*, Studia Logica, (2006) 84(1), 23-50. 1143-1174.
- R. Bělohlávek, *Fuzzy Relational Systems: Foundations and Principles*, Kluwer Academic/Plenum Publishers, New York, 2002.
- R. Bělohlávek, V. Vychodil, *Algebras with fuzzy equalities*, Fuzzy Sets and Systems 157 (2006) 161-201.

Ω-sets and related notions were also applied to foundations of Fuzzy Set Theory, e.g.:

- U. Höhle, *Fuzzy sets and sheaves. Part I: basic concepts*, Fuzzy Sets and Systems, (2007) 158(11),
- S. Gottwald, *Universes of fuzzy sets and axiomatizations of fuzzy set theory, Part II: Category theoretic approaches*, Studia Logica, (2006) 84(1), 23-50. 1143-1174.
- R. Bělohlávek, *Fuzzy Relational Systems: Foundations and Principles*, Kluwer Academic/Plenum Publishers, New York, 2002.
- R. Bělohlávek, V. Vychodil, *Algebras with fuzzy equalities*, Fuzzy Sets and Systems 157 (2006) 161-201.
- M. Demirci, *Foundations of fuzzy functions and vague algebra based on many-valued equivalence relations part I: fuzzy functions and their applications, (part II part III)*, Int. J. General Systems 32 (3) (2003) 123-155, 157-175, 177-201.

- L. Běhounek, U. Bodenhofer, P. Cintula, *Relations in Fuzzy Class Theory: Initial steps*, Fuzzy Sets and Systems 159 (2008) 1729 - 1772.

- L. Běhounek, U. Bodenhofer, P. Cintula, *Relations in Fuzzy Class Theory: Initial steps*, Fuzzy Sets and Systems 159 (2008) 1729 - 1772.
- L. Běhounek, P. Cintula, *Fuzzy class theory*, Fuzzy Sets and Systems 154 (1) (2005) 34-55.

- L. Běhounek, U. Bodenhofer, P. Cintula, *Relations in Fuzzy Class Theory: Initial steps*, Fuzzy Sets and Systems 159 (2008) 1729 - 1772.
- L. Běhounek, P. Cintula, *Fuzzy class theory*, Fuzzy Sets and Systems 154 (1) (2005) 34-55.
- F. Esteva, L. Godo, *Monoidal t-norm based logic: towards a logic for left-continuous t-norms*, Fuzzy Sets and Systems 124 (2001) 271-288.

Lattice-valued functions and relations

Lattice-valued functions and relations

Let Ω be a complete lattice.

Lattice-valued functions and relations

Let Ω be a complete lattice.

A **lattice-valued function** on a nonempty set X is mapping $\mu : X \to \Omega$.

Lattice-valued functions and relations

Let Ω be a complete lattice.

A **lattice-valued function** on a nonempty set X is mapping $\mu : X \to \Omega$.

If $\mu : X \to \Omega$ is a lattice-valued function on a set X then for $p \in \Omega$, the set

$$\mu_p := \{x \in X \mid \mu(x) \geqslant p\}$$

is a ***p*-cut**, or a **cut set**, (**cut**) of μ.

Lattice-valued functions and relations

Let Ω be a complete lattice.

A **lattice-valued function** on a nonempty set X is mapping $\mu : X \to \Omega$.

If $\mu : X \to \Omega$ is a lattice-valued function on a set X then for $p \in \Omega$, the set

$$\mu_p := \{x \in X \mid \mu(x) \geqslant p\}$$

is a ***p*-cut**, or a **cut set**, (**cut**) of μ.
Obviously,

$$\mu_p = \mu^{-1}(\uparrow p).$$

Lattice-valued functions and relations

Let Ω be a complete lattice.

A **lattice-valued function** on a nonempty set X is mapping $\mu : X \to \Omega$.

If $\mu : X \to \Omega$ is a lattice-valued function on a set X then for $p \in \Omega$, the set

$$\mu_p := \{x \in X \mid \mu(x) \geqslant p\}$$

is a ***p*-cut**, or a **cut set**, (**cut**) of μ.
Obviously,

$$\mu_p = \mu^{-1}(\uparrow p).$$

Proposition

The collection $\{\mu_p \mid p \in \Omega\}$ of all cuts of the function $\mu : X \to \Omega$ is a closure system on X.

A **lattice-valued** (binary) **relation** R **on** A is a lattice-valued function on A^2, i.e., it is a mapping $R : A^2 \to \Omega$.

A **lattice-valued** (binary) **relation** R **on** A is a lattice-valued function on A^2, i.e., it is a mapping $R : A^2 \to \Omega$.

R is **symmetric** if

$$R(x, y) = R(y, x) \;\text{ for all }\; x, y \in A;$$

A **lattice-valued** (binary) **relation** R **on** A is a lattice-valued function on A^2, i.e., it is a mapping $R : A^2 \to \Omega$.

R is **symmetric** if

$$R(x,y) = R(y,x) \text{ for all } x, y \in A;$$

R is **transitive** if

$$R(x,y) \geqslant R(x,z) \wedge R(z,y) \text{ for all } x, y, z \in A.$$

Let $\mu : A \to \Omega$ and $R : A^2 \to \Omega$ be a lattice-valued function a lattice-valued relation on A, respectively.

Let $\mu : A \to \Omega$ and $R : A^2 \to \Omega$ be a lattice-valued function a lattice-valued relation on A, respectively.
Then R is a **lattice-valued relation on** μ if for all $x, y \in A$

$$R(x, y) \leqslant \mu(x) \wedge \mu(y).$$

Let $\mu : A \to \Omega$ and $R : A^2 \to \Omega$ be a lattice-valued function a lattice-valued relation on A, respectively.
Then R is a **lattice-valued relation on μ** if for all $x, y \in A$

$$R(x, y) \leqslant \mu(x) \wedge \mu(y).$$

A lattice-valued relation R on $\mu : A \to \Omega$ is said to be **reflexive on μ** or **μ-reflexive** if

$$R(x, x) = \mu(x) \text{ for every } x \in A.$$

Let $\mu : A \to \Omega$ and $R : A^2 \to \Omega$ be a lattice-valued function a lattice-valued relation on A, respectively.
Then R is a **lattice-valued relation on μ** if for all $x, y \in A$

$$R(x,y) \leqslant \mu(x) \wedge \mu(y).$$

A lattice-valued relation R on $\mu : A \to \Omega$ is said to be **reflexive on μ** or **μ-reflexive** if

$$R(x,x) = \mu(x) \text{ for every } x \in A.$$

A symmetric and transitive Ω-valued relation R on A, which is μ-reflexive is a **lattice-valued equivalence** on $\mu : A \to \Omega$.

Let $\mu : A \to \Omega$ and $R : A^2 \to \Omega$ be a lattice-valued function a lattice-valued relation on A, respectively.
Then R is a **lattice-valued relation on μ** if for all $x, y \in A$

$$R(x, y) \leqslant \mu(x) \wedge \mu(y).$$

A lattice-valued relation R on $\mu : A \to \Omega$ is said to be **reflexive on μ** or **μ-reflexive** if

$$R(x, x) = \mu(x) \text{ for every } x \in A.$$

A symmetric and transitive Ω-valued relation R on A, which is μ-reflexive is a **lattice-valued equivalence** on $\mu : A \to \Omega$.

A lattice-valued equivalence R on A fulfills the **strictness** *property:*

$$R(x, y) \leqslant R(x, x) \wedge R(y, y).$$

Let $\mu : A \to \Omega$ and $R : A^2 \to \Omega$ be a lattice-valued function a lattice-valued relation on A, respectively.
Then R is a **lattice-valued relation on μ** if for all $x, y \in A$

$$R(x,y) \leqslant \mu(x) \wedge \mu(y).$$

A lattice-valued relation R on $\mu : A \to \Omega$ is said to be **reflexive on μ** or **μ-reflexive** if

$$R(x,x) = \mu(x) \text{ for every } x \in A.$$

A symmetric and transitive Ω-valued relation R on A, which is μ-reflexive is a **lattice-valued equivalence** on $\mu : A \to \Omega$.

A lattice-valued equivalence R on A fulfills the **strictness** *property:*

$$R(x,y) \leqslant R(x,x) \wedge R(y,y).$$

A lattice-valued equivalence R on A is a **lattice-valued equality**, if it satisfies the **separation property**:

Let $\mu : A \to \Omega$ and $R : A^2 \to \Omega$ be a lattice-valued function a lattice-valued relation on A, respectively.
Then R is a **lattice-valued relation on μ** if for all $x, y \in A$

$$R(x,y) \leqslant \mu(x) \wedge \mu(y).$$

A lattice-valued relation R on $\mu : A \to \Omega$ is said to be **reflexive on μ** or **μ-reflexive** if

$$R(x,x) = \mu(x) \text{ for every } x \in A.$$

A symmetric and transitive Ω-valued relation R on A, which is μ-reflexive is a **lattice-valued equivalence** on $\mu : A \to \Omega$.

A lattice-valued equivalence R on A fulfills the **strictness** *property:*

$$R(x,y) \leqslant R(x,x) \wedge R(y,y).$$

A lattice-valued equivalence R on A is a **lattice-valued equality**, if it satisfies the **separation property**:

$$R(x,y) = R(x,x) = R(y,x) \neq 0 \text{ implies } x = y.$$

If $\mathcal{A} = (A, F)$ is an algebra, then the function $\mu : A \to \Omega$ is **compatible** with the operations on $\mathcal{A}$ if it is not constantly equal to 0, and fulfils the following:

If $\mathcal{A} = (A, F)$ is an algebra, then the function $\mu : A \to \Omega$ is **compatible** with the operations on $\mathcal{A}$ if it is not constantly equal to 0, and fulfils the following:
For any operation f from F with arity greater than 0, $f : A^n \to A, n \in \mathbb{N}$, and for all $a_1, \dots, a_n \in A$, we have that

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(f(a_1, \dots, a_n)),$$

and for a nullary operation $c \in F$, $\mu(c) = 1$.

If $\mathcal{A} = (A, F)$ is an algebra, then the function $\mu : A \to \Omega$ is **compatible** with the operations on $\mathcal{A}$ if it is not constantly equal to 0, and fulfils the following:
For any operation f from F with arity greater than 0, $f : A^n \to A, n \in \mathbb{N}$, and for all $a_1, \ldots, a_n \in A$, we have that

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(f(a_1, \ldots, a_n)),$$

and for a nullary operation $c \in F$, $\mu(c) = 1$.

A lattice-valued relation $R : A^2 \to \Omega$ on an algebra $\mathcal{A} = (A, F)$ is **compatible** with the operations in F if the following holds:

If $\mathcal{A} = (A, F)$ is an algebra, then the function $\mu : A \to \Omega$ is **compatible** with the operations on $\mathcal{A}$ if it is not constantly equal to 0, and fulfils the following:
For any operation f from F with arity greater than 0, $f : A^n \to A, n \in \mathbb{N}$, and for all $a_1, \ldots, a_n \in A$, we have that

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(f(a_1, \ldots, a_n)),$$

and for a nullary operation $c \in F$, $\mu(c) = 1$.

A lattice-valued relation $R : A^2 \to \Omega$ on an algebra $\mathcal{A} = (A, F)$ is **compatible** with the operations in F if the following holds:
For every n-ary operation $f \in F$, for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$, and for every constant (nullary operation) $c \in F$

$$\bigwedge_{i=1}^{n} R(a_i, b_i) \leqslant R(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n));$$

and $R(c, c) = 1$.

Ω-set

Ω-set

Let Ω be a fixed complete lattice.

Ω-set

Let Ω be a fixed complete lattice.
In the sequel we use *'Ω-valued'* instead of *'lattice-valued'*.

Ω-set

Let Ω be a fixed complete lattice.
In the sequel we use *'Ω-valued'* instead of *'lattice-valued'*.

An **Ω-set** is a pair (A, E), where A is a nonempty set, and E is a symmetric and transitive Ω-valued relation on A, fulfilling the separation property.

Ω-set

Let Ω be a fixed complete lattice.
In the sequel we use *'Ω-valued'* instead of *'lattice-valued'*.

An **Ω-set** is a pair (A, E), where A is a nonempty set, and E is a symmetric and transitive Ω-valued relation on A, fulfilling the separation property.

For an Ω-set (A, E), we denote by μ the Ω-valued function on A, defined by

$$\mu(x) := E(x, x).$$

Ω-set

Let Ω be a fixed complete lattice.
In the sequel we use *'Ω-valued'* instead of *'lattice-valued'*.

An **Ω-set** is a pair (A, E), where A is a nonempty set, and E is a symmetric and transitive Ω-valued relation on A, fulfilling the separation property.

For an Ω-set (A, E), we denote by μ the Ω-valued function on A, defined by

$$\mu(x) := E(x, x).$$

We say that μ is *determined by E*.

Ω-set

Let Ω be a fixed complete lattice.
In the sequel we use *'Ω-valued'* instead of *'lattice-valued'*.

An **Ω-set** is a pair (A, E), where A is a nonempty set, and E is a symmetric and transitive Ω-valued relation on A, fulfilling the separation property.

For an Ω-set (A, E), we denote by μ the Ω-valued function on A, defined by

$$\mu(x) := E(x, x).$$

We say that μ is *determined by E*.
By the strictness property, E is an Ω-valued relation on μ, namely, it is an Ω-valued equality on μ.

Proposition

If (A, E) is an Ω-set and $p \in \Omega$, then the cut μ_p is a subset of A, and the cut E_p is an equivalence relation on μ_p.

Proposition

If (A, E) is an Ω-set and $p \in \Omega$, then the cut μ_p is a subset of A, and the cut E_p is an equivalence relation on μ_p.
In addition, the collection of all cuts $\{E_p \mid p \in \Omega\}$ of E is a closure system, a subposet of the lattice of all weak equivalences on A.

Ω-algebra; identities

Ω-algebra; identities

Let $\mathcal{A} = (A, F)$ be an algebra and $E : A^2 \to \Omega$ an Ω-valued equality on A, which is compatible with the operations in F.

Ω-algebra; identities

Let $\mathcal{A} = (A, F)$ be an algebra and $E : A^2 \to \Omega$ an Ω-valued equality on A, which is compatible with the operations in F. Then, $(\mathcal{A}, E)$ is an **Ω-algebra**.

Ω-algebra; identities

Let $\mathcal{A} = (A, F)$ be an algebra and $E : A^2 \to \Omega$ an Ω-valued equality on A, which is compatible with the operations in F.
Then, $(\mathcal{A}, E)$ is an **Ω-algebra**.
Algebra $\mathcal{A}$ is the **underlying algebra** of $(\mathcal{A}, E)$.

Ω-algebra; identities

Let $\mathcal{A} = (A, F)$ be an algebra and $E : A^2 \rightarrow \Omega$ an Ω-valued equality on A, which is compatible with the operations in F.
Then, $(\mathcal{A}, E)$ is an **Ω-algebra**.
Algebra $\mathcal{A}$ is the **underlying algebra** of $(\mathcal{A}, E)$.

The function $\mu : A \rightarrow \Omega$, defined by $\mu(x) = E(x, x)$ is obviously compatible on $\mathcal{A}$.

Ω-algebra; identities

Let $\mathcal{A} = (A, F)$ be an algebra and $E : A^2 \to \Omega$ an Ω-valued equality on A, which is compatible with the operations in F.
Then, $(\mathcal{A}, E)$ is an **Ω-algebra**.
Algebra $\mathcal{A}$ is the **underlying algebra** of $(\mathcal{A}, E)$.

The function $\mu : A \to \Omega$, defined by $\mu(x) = E(x, x)$ is obviously compatible on $\mathcal{A}$.

Proposition

Let $(\mathcal{A}, E)$ be an Ω-algebra. Then the following hold for every $p \in \Omega$:

Ω-algebra; identities

Let $\mathcal{A} = (A, F)$ be an algebra and $E : A^2 \to \Omega$ an Ω-valued equality on A, which is compatible with the operations in F.
Then, $(\mathcal{A}, E)$ is an **Ω-algebra**.
Algebra $\mathcal{A}$ is the **underlying algebra** of $(\mathcal{A}, E)$.

The function $\mu : A \to \Omega$, defined by $\mu(x) = E(x, x)$ is obviously compatible on $\mathcal{A}$.

Proposition

Let $(\mathcal{A}, E)$ be an Ω-algebra. Then the following hold for every $p \in \Omega$:
(i) The cut μ_p of μ is a subalgebra of $\mathcal{A}$, and

Ω-algebra; identities

Let $\mathcal{A} = (A, F)$ be an algebra and $E : A^2 \to \Omega$ an Ω-valued equality on A, which is compatible with the operations in F.
Then, $(\mathcal{A}, E)$ is an **Ω-algebra**.
Algebra $\mathcal{A}$ is the **underlying algebra** of $(\mathcal{A}, E)$.

The function $\mu : A \to \Omega$, defined by $\mu(x) = E(x, x)$ is obviously compatible on $\mathcal{A}$.

Proposition

Let $(\mathcal{A}, E)$ be an Ω-algebra. Then the following hold for every $p \in \Omega$:
(i) The cut μ_p of μ is a subalgebra of $\mathcal{A}$, and
(ii) The cut E_p of E is a congruence relation on μ_p.

Therefore:

Therefore:

Every Ω-algebra $(\mathcal{A}, E)$ uniquely determines a closure system in the lattice $\mathrm{Con_w}(\mathcal{A})$ of weak congruences on $\mathcal{A}$.

Therefore:

Every Ω-algebra $(\mathcal{A}, E)$ uniquely determines a closure system in the lattice $\mathrm{Con_w}(\mathcal{A})$ of weak congruences on $\mathcal{A}$.

The converse:

Therefore:

Every Ω-algebra $(\mathcal{A}, E)$ uniquely determines a closure system in the lattice $\mathrm{Con_w}(\mathcal{A})$ of weak congruences on $\mathcal{A}$.

The converse:

Theorem

Let $\mathcal{A}$ be an algebra and $\mathcal{R}$ a closure system in $\mathrm{Con_w}(\mathcal{A})$ such that for all $a, b \in A$,

$$\text{if } a \neq b, \text{ then } \ (a, b) \notin \bigcap\{R \in \mathcal{R} \mid (a, a) \in R\}.$$

Then there is a complete lattice Ω and an Ω-algebra $(\mathcal{A}, E)$ with the underlying algebra $\mathcal{A}$, such that $\mathcal{R}$ consists of cuts of E.

Let

$$u(x_1, \dots, x_n) \approx v(x_1, \dots, x_n) \quad \text{(briefly} \quad u \approx v)$$

be an identity in the type of an Ω-algebra $(\mathcal{A}, E)$.

Let

$$u(x_1, \ldots, x_n) \approx v(x_1, \ldots, x_n) \quad \text{(briefly} \quad u \approx v\text{)}$$

be an identity in the type of an Ω-algebra $(\mathcal{A}, E)$. We assume that variables appearing in terms u and v are from $x_1, \ldots, x_n$.

Let

$$u(x_1, \ldots, x_n) \approx v(x_1, \ldots, x_n) \quad \text{(briefly } \; u \approx v\text{)}$$

be an identity in the type of an Ω-algebra $(\mathcal{A}, E)$. We assume that variables appearing in terms u and v are from $x_1, \ldots, x_n$.

Then, $(\mathcal{A}, E)$ **satisfies identity** $u \approx v$ (this identity **holds** on $(\mathcal{A}, E)$) if the following condition is fulfilled:

Let

$$u(x_1, \dots, x_n) \approx v(x_1, \dots, x_n) \quad \text{(briefly} \quad u \approx v)$$

be an identity in the type of an Ω-algebra $(\mathcal{A}, E)$. We assume that variables appearing in terms u and v are from $x_1, \dots, x_n$.

Then, $(\mathcal{A}, E)$ **satisfies identity** $u \approx v$ (this identity **holds** on $(\mathcal{A}, E)$) if the following condition is fulfilled:

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant E(u(a_1, \dots, a_n), v(a_1, \dots, a_n)),$$

for all $a_1, \dots, a_n \in A$ and the term-operations u^A and v^A on $\mathcal{A}$ corresponding to terms u and v respectively.

If Ω-algebra $(\mathcal{A}, E)$ satisfies an identity, then this identity does not necessarily hold on $\mathcal{A}$.

If Ω-algebra $(\mathcal{A}, E)$ satisfies an identity, then this identity does not necessarily hold on $\mathcal{A}$.
On the other hand, if the supporting algebra fulfills an identity then also the corresponding Ω-algebra does.

If Ω-algebra $(\mathcal{A}, E)$ satisfies an identity, then this identity does not necessarily hold on $\mathcal{A}$.
On the other hand, if the supporting algebra fulfills an identity then also the corresponding Ω-algebra does.

Proposition

If an identity $u \approx v$ holds on an algebra $\mathcal{A}$, then it also holds on an Ω-algebra $(\mathcal{A}, E)$.

Theorem

Let $(\mathcal{A}, E)$ be an Ω-algebra, and $\mathcal{F}$ a set of identities in the language of $\mathcal{A}$. Then, $(\mathcal{A}, E)$ satisfies all identities in $\mathcal{F}$ if and only if for every $p \in L$ the quotient algebra μ_p/E_p satisfies the same identities.

Theorem

Let $(\mathcal{A}, E)$ be an Ω-algebra, and $\mathcal{F}$ a set of identities in the language of $\mathcal{A}$. Then, $(\mathcal{A}, E)$ satisfies all identities in $\mathcal{F}$ if and only if for every $p \in L$ the quotient algebra μ_p/E_p satisfies the same identities.
In addition, the poset

$$(\{\mu_p/E_p \mid p \in \Omega\}, \subseteq)$$

is a closure system which is, up to an isomorphism, a subposet of the weak congruence lattice of $\mathcal{A}$.

Ω-subalgebra

Ω-subalgebra

Let $(\mathcal{A}, E)$ be an Ω-algebra, and $E_1 : A \to \Omega$ a symmetric and transitive Ω-relation on A, so that the following holds:

Ω-subalgebra

Let $(\mathcal{A}, E)$ be an Ω-algebra, and $E_1 : A \to \Omega$ a symmetric and transitive Ω-relation on A, so that the following holds:
For all $x, y \in A$

$$E_1(x,y) = E(x,y) \wedge E_1(x,x) \wedge E_1(y,y).$$

Ω-subalgebra

Let $(\mathcal{A}, E)$ be an Ω-algebra, and $E_1 : A \to \Omega$ a symmetric and transitive Ω-relation on A, so that the following holds:
For all $x, y \in A$

$$E_1(x, y) = E(x, y) \wedge E_1(x, x) \wedge E_1(y, y).$$

Let also E_1 be compatible with the operations in $\mathcal{A}$.

Ω-subalgebra

Let $(\mathcal{A}, E)$ be an Ω-algebra, and $E_1 : A \to \Omega$ a symmetric and transitive Ω-relation on A, so that the following holds:
For all $x, y \in A$

$$E_1(x, y) = E(x, y) \wedge E_1(x, x) \wedge E_1(y, y).$$

Let also E_1 be compatible with the operations in $\mathcal{A}$.

Obviously, $(\mathcal{A}, E_1)$ is an Ω-algebra and we say that it is an **Ω-subalgebra** of $(\mathcal{A}, E)$.

Ω-subalgebra

Let $(\mathcal{A}, E)$ be an Ω-algebra, and $E_1 : A \to \Omega$ a symmetric and transitive Ω-relation on A, so that the following holds:
For all $x, y \in A$

$$E_1(x, y) = E(x, y) \wedge E_1(x, x) \wedge E_1(y, y).$$

Let also E_1 be compatible with the operations in $\mathcal{A}$.

Obviously, $(\mathcal{A}, E_1)$ is an Ω-algebra and we say that it is an **Ω-subalgebra** of $(\mathcal{A}, E)$.

Proposition

If $(\mathcal{A}, E_1)$ is an Ω-subalgebra of an Ω-algebra $(\mathcal{A}, E)$, and $\mu_1 : A \to \Omega$ is an Ω-valued function on A, defined by $\mu_1(x) := E_1(x, x)$, then μ_1 is compatible on $\mathcal{A}$.

An Ω-subalgebra $(\mathcal{A}, E_1)$ of $(\mathcal{A}, E)$ fulfills all the identities that the latter does:

An Ω-subalgebra $(\mathcal{A}, E_1)$ of $(\mathcal{A}, E)$ fulfills all the identities that the latter does:

Theorem

Let $(\mathcal{A}, E_1)$ be an Ω-subalgebra of an Ω-algebra $(\mathcal{A}, E)$. If $(\mathcal{A}, E)$ satisfies the set Σ of identities, then also $(\mathcal{A}, E_1)$ satisfies all identities in Σ.

Example: Ω-group

Example: Ω-group

Let $(\mathcal{G}, E)$ be an Ω-algebra in which $\mathcal{G} = (G, \cdot, ^{-1}, e)$ is an algebra with a binary operation $(\,\cdot\,)$, unary operation $(^{-1})$ and a constant (e).

Example: Ω-group

Let $(\mathcal{G}, E)$ be an Ω-algebra in which $\mathcal{G} = (G, \cdot, ^{-1}, e)$ is an algebra with a binary operation $(\,\cdot\,)$, unary operation $(^{-1})$ and a constant (e).

Then, $(\mathcal{G}, E)$ is an **Ω-group** if the following known group identities hold with respect to E:

Example: Ω-group

Let $(\mathcal{G}, E)$ be an Ω-algebra in which $\mathcal{G} = (G, \cdot, ^{-1}, e)$ is an algebra with a binary operation $(\cdot)$, unary operation $(^{-1})$ and a constant (e).

Then, $(\mathcal{G}, E)$ is an **Ω-group** if the following known group identities hold with respect to E:

$$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z,$$

$$x \cdot e \approx x, \; e \cdot x \approx x,$$

$$x \cdot x^{-1} \approx e, \; x^{-1} \cdot x \approx e.$$

Example: Ω-group

Let $(\mathcal{G}, E)$ be an Ω-algebra in which $\mathcal{G} = (G, \cdot, ^{-1}, e)$ is an algebra with a binary operation $(\cdot)$, unary operation $(^{-1})$ and a constant (e).

Then, $(\mathcal{G}, E)$ is an **Ω-group** if the following known group identities hold with respect to E:

$$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z,$$

$$x \cdot e \approx x, \; e \cdot x \approx x,$$

$$x \cdot x^{-1} \approx e, \; x^{-1} \cdot x \approx e.$$

In terms of Ω-algebras, these identities are equivalent with formulas:

Example: Ω-group

Let $(\mathcal{G}, E)$ be an Ω-algebra in which $\mathcal{G} = (G, \cdot, ^{-1}, e)$ is an algebra with a binary operation $(\cdot)$, unary operation $(^{-1})$ and a constant (e).

Then, $(\mathcal{G}, E)$ is an **Ω-group** if the following known group identities hold with respect to E:

$$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z,$$
$$x \cdot e \approx x, \; e \cdot x \approx x,$$
$$x \cdot x^{-1} \approx e, \; x^{-1} \cdot x \approx e.$$

In terms of Ω-algebras, these identities are equivalent with formulas:

(*i*) $E(x \cdot (y \cdot z), (x \cdot y) \cdot z) \geqslant \mu(x) \wedge \mu(y) \wedge \mu(z)$,

(*ii*) $E(x \cdot e, x) \geqslant \mu(x)$ and $E(e \cdot x, x) \geqslant \mu(x)$,

(*iii*) $E(x \cdot x^{-1}, e) \geqslant \mu(x)$ and $E(x^{-1} \cdot x, e) \geqslant \mu(x)$.

Theorem

Let $(\mathcal{G}, E)$ be an Ω-algebra. Then, $(\mathcal{G}, E)$ is an Ω-group if and only if for every $p \in \Omega$, the cut μ_p is a subalgebra of $\mathcal{G}$, the cut relation E_p is a congruence on μ_p, and the quotient structure μ_p/E_p is a group.

Normal Ω-subgroup

Normal Ω-subgroup

Let $\overline{\mathcal{G}} = (\mathcal{G}, E^{\mu})$ be an Ω-group and $\overline{\mathcal{N}} = (\mathcal{G}, E^{\nu})$ an Ω-subgroup of $\overline{\mathcal{G}}$.

Normal Ω-subgroup

Let $\overline{\mathcal{G}} = (\mathcal{G}, E^{\mu})$ be an Ω-group and $\overline{\mathcal{N}} = (\mathcal{G}, E^{\nu})$ an Ω-subgroup of $\overline{\mathcal{G}}$.

Then, $\overline{\mathcal{N}}$ is a **normal Ω-subgroup** of $\overline{\mathcal{G}}$, if there is an Ω-valued congruence Θ on $\overline{\mathcal{G}}$, such that for all $x, y \in G$,

$$E^{\nu}(x, y) = E^{\mu}(x, y) \wedge \Theta(e, x) \wedge \Theta(e, y).$$

Normal Ω-subgroup

Let $\overline{\mathcal{G}} = (\mathcal{G}, E^{\mu})$ be an Ω-group and $\overline{\mathcal{N}} = (\mathcal{G}, E^{\nu})$ an Ω-subgroup of $\overline{\mathcal{G}}$.

Then, $\overline{\mathcal{N}}$ is a **normal Ω-subgroup** of $\overline{\mathcal{G}}$, if there is an Ω-valued congruence Θ on $\overline{\mathcal{G}}$, such that for all $x, y \in G$,

$$E^{\nu}(x, y) = E^{\mu}(x, y) \wedge \Theta(e, x) \wedge \Theta(e, y).$$

Theorem

An Ω-subgroup $\overline{\mathcal{N}} = (\mathcal{G}, E^{\nu})$ of an Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E^{\mu})$ is a normal Ω-subgroup of $\overline{\mathcal{G}}$, if and only if for every $p \in \Omega$, ν_p / E^{ν}_p is a normal subgroup of the group μ_p / E^{μ}_p.

Normal Ω-subgroup

Let $\overline{\mathcal{G}} = (\mathcal{G}, E^{\mu})$ be an Ω-group and $\overline{\mathcal{N}} = (\mathcal{G}, E^{\nu})$ an Ω-subgroup of $\overline{\mathcal{G}}$.

Then, $\overline{\mathcal{N}}$ is a **normal Ω-subgroup** of $\overline{\mathcal{G}}$, if there is an Ω-valued congruence Θ on $\overline{\mathcal{G}}$, such that for all $x, y \in G$,

$$E^{\nu}(x, y) = E^{\mu}(x, y) \wedge \Theta(e, x) \wedge \Theta(e, y).$$

Theorem

An Ω-subgroup $\overline{\mathcal{N}} = (\mathcal{G}, E^{\nu})$ of an Ω-group $\overline{\mathcal{G}} = (\mathcal{G}, E^{\mu})$ is a normal Ω-subgroup of $\overline{\mathcal{G}}$, if and only if for every $p \in \Omega$, ν_p / E_p^{ν} is a normal subgroup of the group μ_p / E_p^{μ}.

Corollary

If $\overline{\mathcal{G}} = (\mathcal{G}, E^{\mu})$ is a commutative Ω-group, then every Ω-subgroup of $\overline{\mathcal{G}}$ is normal.

Concrete example

Concrete example

$\mathcal{G} = (\mathbb{N}_0, \oplus, ^{-1}, 0)$,

Concrete example

$\mathcal{G} = (\mathbb{N}_0, \oplus, ^{-1}, 0), \quad \mathbb{N}_0 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots\}$

Concrete example

$\mathcal{G} = (\mathbb{N}_0, \oplus, ^{-1}, 0), \quad \mathbb{N}_0 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots\}$

$\oplus$ – a binary operation on $\mathbb{N}_0$:

Concrete example

$\mathcal{G} = (\mathbb{N}_0, \oplus, ^{-1}, 0), \quad \mathbb{N}_0 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots\}$

$\oplus$ – a binary operation on $\mathbb{N}_0$:

$$x \oplus y := \begin{cases} \mathbf{0} & \text{if} \quad x = y \\ x + y & \text{if} \quad x \neq y \end{cases},$$

Concrete example

$\mathcal{G} = (\mathbb{N}_0, \oplus, ^{-1}, 0), \quad \mathbb{N}_0 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots\}$

$\oplus$ – a binary operation on $\mathbb{N}_0$:

$$x \oplus y := \begin{cases} \mathbf{0} & \text{if} \quad x = y \\ x + y & \text{if} \quad x \neq y \end{cases},$$

$^{-1}$ – a unary operation on $\mathbb{N}_0$ defined by $x^{-1} = x$.

Concrete example

$\mathcal{G} = (\mathbb{N}_0, \oplus, ^{-1}, 0), \quad \mathbb{N}_0 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots\}$

$\oplus$ – a binary operation on $\mathbb{N}_0$:

$$x \oplus y := \begin{cases} \mathbf{0} & \text{if} \quad x = y \\ x + y & \text{if} \quad x \neq y \end{cases},$$

$^{-1}$ – a unary operation on $\mathbb{N}_0$ defined by $x^{-1} = x$.

A neutral element in $\mathcal{G}$ is 0, but $\oplus$ is not associative, hence $\mathcal{G}$ is not a group.

Concrete example

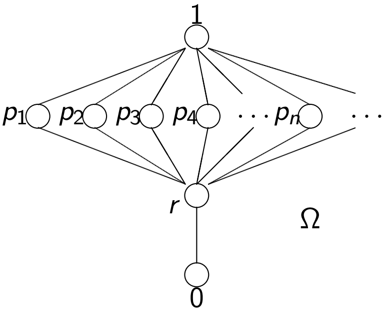
$\mathcal{G} = (\mathbb{N}_0, \oplus, ^{-1}, 0)$, $\quad \mathbb{N}_0 = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots\}$

$\oplus$ – a binary operation on $\mathbb{N}_0$:

$$x \oplus y := \begin{cases} \mathbf{0} & \text{if} \quad x = y \\ x + y & \text{if} \quad x \neq y \end{cases},$$

$^{-1}$ – a unary operation on $\mathbb{N}_0$ defined by $x^{-1} = x$.

A neutral element in $\mathcal{G}$ is 0, but $\oplus$ is not associative, hence $\mathcal{G}$ is not a group.



$$\mu := \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \dots & \mathbf{n} & \dots \\ 1 & p_1 & p_2 & p_3 & \dots & p_n & \dots \end{pmatrix}.$$

$$\mu := \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \dots & \mathbf{n} & \dots \\ 1 & p_1 & p_2 & p_3 & \dots & p_n & \dots \end{pmatrix}.$$

E^μ	**0**	**1**	**2**	**3**	**4**	**5**	$\dots$
0	1	0	r	0	r	0	$\cdots$
1	0	p_1	0	r	0	r	$\cdots$
2	r	0	p_2	0	r	0	$\cdots$
3	0	r	0	p_3	0	r	$\cdots$
4	r	0	r	0	p_4	0	$\cdots$
5	0	r	0	r	0	p_5	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

$$\mu := \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \dots & \mathbf{n} & \dots \\ 1 & p_1 & p_2 & p_3 & \dots & p_n & \dots \end{pmatrix}.$$

E^μ	**0**	**1**	**2**	**3**	**4**	**5**	...
0	1	0	r	0	r	0	...
1	0	p_1	0	r	0	r	...
2	r	0	p_2	0	r	0	...
3	0	r	0	p_3	0	r	...
4	r	0	r	0	p_4	0	...
5	0	r	0	r	0	p_5	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	

The structure $(\mathcal{G}, E^\mu)$ is an Ω-group.

Cut subalgebras:

Cut subalgebras:

μ_1 – the trivial one-element subalgebra $\{\mathbf{0}\}$.

Cut subalgebras:

μ_1 – the trivial one-element subalgebra $\{\mathbf{0}\}$.
For every $p_n \in \Omega$, $\mu_{p_n} = \{\mathbf{0}, \mathbf{n}\}$.

Cut subalgebras:

μ_1 – the trivial one-element subalgebra $\{\mathbf{0}\}$.
For every $p_n \in \Omega$, $\mu_{p_n} = \{\mathbf{0}, \mathbf{n}\}$.

$\oplus$	$\mathbf{0}$	$\mathbf{n}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{n}$
$\mathbf{n}$	$\mathbf{n}$	$\mathbf{0}$

;

$E^{\mu}_{p_n}$	$\mathbf{0}$	$\mathbf{n}$
$\mathbf{0}$	1	0
$\mathbf{n}$	0	1

.

Cut subalgebras:

μ_1 – the trivial one-element subalgebra $\{\mathbf{0}\}$.
For every $p_n \in \Omega$, $\mu_{p_n} = \{\mathbf{0}, \mathbf{n}\}$.

$\oplus$	$\mathbf{0}$	$\mathbf{n}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{n}$
$\mathbf{n}$	$\mathbf{n}$	$\mathbf{0}$

;

$E^{\mu}_{p_n}$	$\mathbf{0}$	$\mathbf{n}$
$\mathbf{0}$	1	0
$\mathbf{n}$	0	1

.

For every $p_n \in \Omega$, the quotient structure $\mu_{p_n}/E^{\mu}_{p_n}$ is a two-element group.

Relational structures: Ω-poset and Ω-lattice

Relational structures: Ω-poset and Ω-lattice

Let E be an Ω-valued equality on a nonempty set A.

Relational structures: Ω-poset and Ω-lattice

Let E be an Ω-valued equality on a nonempty set A.
An Ω-valued relation $R : A^2 \to \Omega$ on A is ***E*-antisymmetric**, if the following holds:

$$R(x, y) \wedge R(y, x) = E(x, y), \quad \text{for all} \quad x, y \in A.$$

Relational structures: Ω-poset and Ω-lattice

Let E be an Ω-valued equality on a nonempty set A.
An Ω-valued relation $R : A^2 \to \Omega$ on A is **E-antisymmetric**, if the following holds:

$$R(x, y) \wedge R(y, x) = E(x, y), \quad \text{for all} \quad x, y \in A.$$

Let (M, E) be an Ω-set.

Relational structures: Ω-poset and Ω-lattice

Let E be an Ω-valued equality on a nonempty set A.
An Ω-valued relation $R : A^2 \to \Omega$ on A is **E-antisymmetric**, if the following holds:

$$R(x,y) \wedge R(y,x) = E(x,y), \quad \text{for all} \quad x, y \in A.$$

Let (M, E) be an Ω-set.
An Ω-valued relation $R : M^2 \to \Omega$ on M is an **Ω-valued order** on (M, E), if it fulfills the strictness property:

$$R(x,y) \leqslant R(x,x) \wedge R(y,y),$$

it is E-antisymmetric, and it is transitive:

$$R(x,z) \wedge R(z,y) \leqslant R(x,y) \text{ for all } x, y, z \in M.$$

Relational structures: Ω-poset and Ω-lattice

Let E be an Ω-valued equality on a nonempty set A.
An Ω-valued relation $R : A^2 \to \Omega$ on A is **E-antisymmetric**, if the following holds:

$$R(x,y) \wedge R(y,x) = E(x,y), \quad \text{for all} \quad x,y \in A.$$

Let (M,E) be an Ω-set.
An Ω-valued relation $R : M^2 \to \Omega$ on M is an **Ω-valued order** on (M,E), if it fulfills the strictness property:

$$R(x,y) \leqslant R(x,x) \wedge R(y,y),$$

it is E-antisymmetric, and it is transitive:

$$R(x,z) \wedge R(z,y) \leqslant R(x,y) \text{ for all } x,y,z \in M.$$

A structure (M,E,R) is an **Ω-poset**, if (M,E) is an Ω-set, and $R : M^2 \to \Omega$ is an Ω-valued order on (M,E).

It is clear that by E-antisymmetry, $R(x,x) = E(x,x)$, for every $x \in M$.

It is clear that by E-antisymmetry, $R(x,x) = E(x,x)$, for every $x \in M$.

As usual, we denote by μ the Ω-valued function on M, defined by $\mu(x) = E(x,x) = R(x,x)$.

It is clear that by E-antisymmetry, $R(x, x) = E(x, x)$, for every $x \in M$.
As usual, we denote by μ the Ω-valued function on M, defined by $\mu(x) = E(x, x) = R(x, x)$.

In an Ω-set, every cut E_p of E is a classical equivalence relation on the cut μ_p of μ.

It is clear that by E-antisymmetry, $R(x, x) = E(x, x)$, for every $x \in M$.
As usual, we denote by μ the Ω-valued function on M, defined by $\mu(x) = E(x, x) = R(x, x)$.

In an Ω-set, every cut E_p of E is a classical equivalence relation on the cut μ_p of μ.
Let $[x]_{E_p}$ be the equivalence class of $x \in \mu_p$.

It is clear that by E-antisymmetry, $R(x,x) = E(x,x)$, for every $x \in M$.
As usual, we denote by μ the Ω-valued function on M, defined by $\mu(x) = E(x,x) = R(x,x)$.

In an Ω-set, every cut E_p of E is a classical equivalence relation on the cut μ_p of μ.
Let $[x]_{E_p}$ be the equivalence class of $x \in \mu_p$.
μ_p/E_p is the corresponding quotient set: for $p \in \Omega$
$[x]_{E_p} := \{y \in \mu_p \mid xE_py\}, x \in \mu_p; \;\; \mu_p/E_p := \{[x]_{E_p} \mid x \in \mu_p\}$.

It is clear that by E-antisymmetry, $R(x,x) = E(x,x)$, for every $x \in M$.
As usual, we denote by μ the Ω-valued function on M, defined by $\mu(x) = E(x,x) = R(x,x)$.

In an Ω-set, every cut E_p of E is a classical equivalence relation on the cut μ_p of μ.
Let $[x]_{E_p}$ be the equivalence class of $x \in \mu_p$.
μ_p/E_p is the corresponding quotient set: for $p \in \Omega$
$[x]_{E_p} := \{y \in \mu_p \mid xE_p y\}, x \in \mu_p;\;\; \mu_p/E_p := \{[x]_{E_p} \mid x \in \mu_p\}$.

Proposition

Let (M, E, R) be an Ω-poset. Then for every $p \in \Omega$, the binary relation $\leq_p$ on μ_p/E_p, defined by
$[x]_{E_p} \leq_p [y]_{E_p}$ *if and only if* $(x,y) \in R_p$
is a classic ordering relation.

Let (M, E, R) be an Ω-poset and $a, b \in M$. An element $c \in M$ is a **pseudo-infimum** of a and b, if for every $p \leqslant \mu(a) \wedge \mu(b)$ the following holds:

Let (M, E, R) be an Ω-poset and $a, b \in M$. An element $c \in M$ is a **pseudo-infimum** of a and b, if for every $p \leqslant \mu(a) \wedge \mu(b)$ the following holds:

(i) $p \leqslant R(c, a) \wedge R(c, b)$ and

for every $x \in \mu_p$

$p \leqslant R(x, a) \wedge R(x, b)$ implies $p \leqslant R(x, c)$.

Let (M, E, R) be an Ω-poset and $a, b \in M$. An element $c \in M$ is a **pseudo-infimum** of a and b, if for every $p \leqslant \mu(a) \wedge \mu(b)$ the following holds:
(i) $p \leqslant R(c, a) \wedge R(c, b)$ and
for every $x \in \mu_p$
$p \leqslant R(x, a) \wedge R(x, b)$ implies $p \leqslant R(x, c)$.

An element $d \in M$ is a **pseudo-supremum** of $a, b \in M$, if for every $p \leqslant \mu(a) \wedge \mu(b)$ the following holds:

Let (M, E, R) be an Ω-poset and $a, b \in M$. An element $c \in M$ is a **pseudo-infimum** of a and b, if for every $p \leqslant \mu(a) \wedge \mu(b)$ the following holds:

(i) $p \leqslant R(c, a) \wedge R(c, b)$ and

for every $x \in \mu_p$

$p \leqslant R(x, a) \wedge R(x, b)$ implies $p \leqslant R(x, c)$.

An element $d \in M$ is a **pseudo-supremum** of $a, b \in M$, if for every $p \leqslant \mu(a) \wedge \mu(b)$ the following holds:

(ii) $p \leqslant R(a, d) \wedge R(b, d)$ and

for every $x \in \mu_p$

$p \leqslant R(a, x) \wedge R(b, x)$ implies $p \leqslant R(d, x)$.

Let (M, E, R) be an Ω-poset and $a, b \in M$. An element $c \in M$ is a **pseudo-infimum** of a and b, if for every $p \leqslant \mu(a) \wedge \mu(b)$ the following holds:
(i) $p \leqslant R(c, a) \wedge R(c, b)$ and
for every $x \in \mu_p$
$p \leqslant R(x, a) \wedge R(x, b)$ implies $p \leqslant R(x, c)$.

An element $d \in M$ is a **pseudo-supremum** of $a, b \in M$, if for every $p \leqslant \mu(a) \wedge \mu(b)$ the following holds:
(ii) $p \leqslant R(a, d) \wedge R(b, d)$ and
for every $x \in \mu_p$
$p \leqslant R(a, x) \wedge R(b, x)$ implies $p \leqslant R(d, x)$.

It is straightforward that a pseudo-infimum (supremum) of a and b belongs to μ_p for every $p \leqslant \mu(a) \wedge \mu(b)$.

A pseudo-infimum and a pseudo-supremum for given $a, b \in M$, if they exist, are not unique in general.

A pseudo-infimum and a pseudo-supremum for given $a, b \in M$, if they exist, are not unique in general.

Proposition

Let (M, E, R) be an Ω-poset and $a, b, c, c_1, d, d_1 \in M$. If c is a pseudo-infimum of a and b, then $\mu(a) \wedge \mu(b) \leqslant E(c, c_1)$ if and only if c_1 is also a pseudo-infimum of a and b. Analogously, if d is a pseudo-supremum of a and b, then $\mu(a) \wedge \mu(b) \leqslant E(d, d_1)$ if and only if d_1 is also a pseudo-supremum of a and b.

A pseudo-infimum and a pseudo-supremum for given $a, b \in M$, if they exist, are not unique in general.

Proposition

Let (M, E, R) be an Ω-poset and $a, b, c, c_1, d, d_1 \in M$. If c is a pseudo-infimum of a and b, then $\mu(a) \wedge \mu(b) \leqslant E(c, c_1)$ if and only if c_1 is also a pseudo-infimum of a and b. Analogously, if d is a pseudo-supremum of a and b, then $\mu(a) \wedge \mu(b) \leqslant E(d, d_1)$ if and only if d_1 is also a pseudo-supremum of a and b.

Since for $p \leqslant q$, every equivalence class of μ_q/E_q is contained in a class of μ_p/E_p, we get that pseudo-infima (suprema) of two elements a, b, if they exist, belong to the same equivalence class in μ_p/E_p, for $p \leqslant \mu(a) \wedge \mu(b)$.

We say that an Ω-poset (M, E, R) is an **Ω-lattice as an ordered structure**, if for every $a, b \in M$ there exist a pseudo-infimum and a pseudo-supremum.

We say that an Ω-poset (M, E, R) is an **Ω-lattice as an ordered structure**, if for every $a, b \in M$ there exist a pseudo-infimum and a pseudo-supremum.

Theorem

Let (M, E, R) be an Ω-poset. Then it is an Ω-lattice as an ordered structure if and only if for every $q \in \Omega$, the poset $(\mu_q / E_q, \leq_q)$ is a lattice, and the following holds:
for all $a, b \in M$, and $p = \mu(a) \wedge \mu(b)$,
$\inf([a]_{E_p}, [b]_{E_p}) \subseteq \inf([a]_{E_q}, [b]_{E_q})$ *and*
$\sup([a]_{E_p}, [b]_{E_p}) \subseteq \sup([a]_{E_q}, [b]_{E_q})$,
for every q, $q \leqslant p$.

Ω-lattice as Ω-algebra

Ω-lattice as Ω-algebra

Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid and $E : M^2 \to \Omega$ an Ω-valued equality on M, hence (M, E) is supposed to be an Ω-set.

Ω-lattice as Ω-algebra

Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid and $E : M^2 \to \Omega$ an Ω-valued equality on M, hence (M, E) is supposed to be an Ω-set.
In addition, E should be compatible with operations $\sqcap$ and $\sqcup$ in the following sense:

Ω-lattice as Ω-algebra

Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid and $E : M^2 \to \Omega$ an Ω-valued equality on M, hence (M, E) is supposed to be an Ω-set.
In addition, E should be compatible with operations $\sqcap$ and $\sqcup$ in the following sense:
$E(x, y) \wedge E(z, t) \leqslant E(x \sqcap z, y \sqcap t)$ and
$E(x, y) \wedge E(z, t) \leqslant E(x \sqcup z, y \sqcup t)$.

Ω-lattice as Ω-algebra

Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid and $E : M^2 \to \Omega$ an Ω-valued equality on M, hence (M, E) is supposed to be an Ω-set.
In addition, E should be compatible with operations $\sqcap$ and $\sqcup$ in the following sense:
$E(x, y) \wedge E(z, t) \leqslant E(x \sqcap z, y \sqcap t)$ and
$E(x, y) \wedge E(z, t) \leqslant E(x \sqcup z, y \sqcup t)$.

Proposition

If E is a compatible Ω-valued equality on a bi-groupoid $\mathcal{M} = (M, \sqcap, \sqcup)$, and $\mu : M \to \Omega$ is defined by $\mu(x) = E(x, x)$, then the following hold:
(i) For all $x, y \in M$,
$\mu(x) \wedge \mu(y) \leqslant \mu(x \sqcap y)$ and $\mu(x) \wedge \mu(y) \leqslant \mu(x \sqcup y)$.
(ii) For every $p \in \Omega$, the cut μ_p of μ is a sub-bi-groupoid of $\mathcal{M}$.
(iii) For every $p \in \Omega$, the cut E_p of E is a congruence on μ_p.

Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid and $(\mathcal{M}, E)$ an Ω-algebra. Then $(\mathcal{M}, E)$ is an **Ω-lattice as an Ω-algebra** (Ω-lattice as an algebra), if it satisfies the lattice identities:

Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid and $(\mathcal{M}, E)$ an Ω-algebra. Then $(\mathcal{M}, E)$ is an **Ω-lattice as an Ω-algebra** (Ω-lattice as an algebra), if it satisfies the lattice identities:

$$\begin{array}{ll}
\ell 1: \; x \sqcap y \approx y \sqcap x & \\
\ell 2: \; x \sqcup y \approx y \sqcup x & (commutativity) \\
\ell 3: \; x \sqcap (y \sqcap z) \approx (x \sqcap y) \sqcap z & \\
\ell 4: \; x \sqcup (y \sqcup z) \approx (x \sqcup y) \sqcup z & (associativity) \\
\ell 5: \; (x \sqcap y) \sqcup x \approx x & \\
\ell 6: \; (x \sqcup y) \sqcap x \approx x. & (absorption)
\end{array}$$

In terms of Ω-algebras for all $x, y, z \in M$, the following formulas should be satisfied, where, as already indicated, the mapping $\mu : M \to \Omega$ is defined by $\mu(x) = E(x, x)$:

In terms of Ω-algebras for all $x, y, z \in M$, the following formulas should be satisfied, where, as already indicated, the mapping $\mu : M \to \Omega$ is defined by $\mu(x) = E(x, x)$:

$$\begin{aligned}
L1 &: \ \mu(x) \wedge \mu(y) \leqslant E(x \sqcap y, y \sqcap x) \\
L2 &: \ \mu(x) \wedge \mu(y) \leqslant E(x \sqcup y, y \sqcup x) \\
L3 &: \ \mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E((x \sqcap y) \sqcap z, x \sqcap (y \sqcap z)) \\
L4 &: \ \mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E((x \sqcup y) \sqcup z, x \sqcup (y \sqcup z)) \\
L5 &: \ \mu(x) \wedge \mu(y) \leqslant E((x \sqcap y) \sqcup x, x) \\
L6 &: \ \mu(x) \wedge \mu(y) \leqslant E((x \sqcup y) \sqcap x, x).
\end{aligned}$$

In terms of Ω-algebras for all $x, y, z \in M$, the following formulas should be satisfied, where, as already indicated, the mapping $\mu : M \to \Omega$ is defined by $\mu(x) = E(x, x)$:

$$
\begin{aligned}
L1 &: \ \mu(x) \wedge \mu(y) \leqslant E(x \sqcap y, y \sqcap x) \\
L2 &: \ \mu(x) \wedge \mu(y) \leqslant E(x \sqcup y, y \sqcup x) \\
L3 &: \ \mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E((x \sqcap y) \sqcap z, x \sqcap (y \sqcap z)) \\
L4 &: \ \mu(x) \wedge \mu(y) \wedge \mu(z) \leqslant E((x \sqcup y) \sqcup z, x \sqcup (y \sqcup z)) \\
L5 &: \ \mu(x) \wedge \mu(y) \leqslant E((x \sqcap y) \sqcup x, x) \\
L6 &: \ \mu(x) \wedge \mu(y) \leqslant E((x \sqcup y) \sqcap x, x).
\end{aligned}
$$

Theorem

Let $\mathcal{M} = (M, \sqcap, \sqcup)$ be a bi-groupoid, and let E be an Ω-valued compatible equality on $\mathcal{M}$. Then, $(\mathcal{M}, E)$ is an Ω-lattice if and only if for every $p \in \Omega$, the quotient structure μ_p / E_p is a lattice.

Equivalence of two approaches

Equivalence of two approaches

Let (M, E, R) be an Ω-lattice as an ordered structure.

Equivalence of two approaches

Let (M, E, R) be an Ω-lattice as an ordered structure.
Using Axiom of Choice, we define two binary operations, $\sqcap$ and $\sqcup$ on M as follows:

Equivalence of two approaches

Let (M, E, R) be an Ω-lattice as an ordered structure.
Using Axiom of Choice, we define two binary operations, $\sqcap$ and $\sqcup$ on M as follows:

For every pair a, b of elements from M, $a \sqcap b$ is an arbitrary, fixed pseudo-infimum of a and b, and $a \sqcup b$ is an arbitrary, fixed pseudo-supremum of a and b.

Equivalence of two approaches

Let (M, E, R) be an Ω-lattice as an ordered structure.
Using Axiom of Choice, we define two binary operations, $\sqcap$ and $\sqcup$ on M as follows:

For every pair a, b of elements from M, $a \sqcap b$ is an arbitrary, fixed pseudo-infimum of a and b, and $a \sqcup b$ is an arbitrary, fixed pseudo-supremum of a and b.

Theorem

If (M, E, R) is an Ω-lattice as an ordered structure, and $\mathcal{M} = (M, \sqcap, \sqcup)$ the bi-groupoid in which operations $\sqcap$, $\sqcup$ are introduced above, then $(\mathcal{M}, E)$ is an Ω-lattice as an algebra.

Theorem

Let $\mathcal{M} = (M, \sqcap, \sqcup)$ *be a bi-groupoid,* $(\mathcal{M}, E)$ *an* Ω*-lattice as an algebra and* $R : M^2 \to \Omega$ *an* Ω*-valued relation on* M *defined by*
$R(x, y) := \mu(x) \wedge \mu(y) \wedge E(x \sqcap y, x)$.
Then, (M, E, R) *is an* Ω*-lattice as an ordered structure.*

Example

Example

Let $M = \{a, b, c, d, e, f, g\}$, and let Ω be the lattice given in Figure 1.

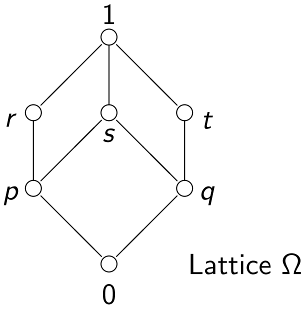


Figure 1

E	a	b	c	d	e	f	g
a	r	p	0	0	0	0	0
b	p	r	0	0	0	0	0
c	0	0	s	q	q	0	0
d	0	0	q	1	q	0	0
e	0	0	q	q	1	0	0
f	0	0	0	0	0	q	0
g	0	0	0	0	0	0	q

Table 1: Ω-valued equality E

R	a	b	c	d	e	f	g
a	r	r	0	0	r	0	0
b	p	r	0	0	r	0	0
c	0	0	s	q	s	q	q
d	r	r	s	1	1	q	q
e	0	0	q	q	1	q	q
f	0	0	0	0	0	q	q
g	0	0	0	0	0	0	q

Table 2: Ω-valued order R

E	a	b	c	d	e	f	g
a	r	p	0	0	0	0	0
b	p	r	0	0	0	0	0
c	0	0	s	q	q	0	0
d	0	0	q	1	q	0	0
e	0	0	q	q	1	0	0
f	0	0	0	0	0	q	0
g	0	0	0	0	0	0	q

Table 1: Ω-valued equality E

R	a	b	c	d	e	f	g
a	r	r	0	0	r	0	0
b	p	r	0	0	r	0	0
c	0	0	s	q	s	q	q
d	r	r	s	1	1	q	q
e	0	0	q	q	1	q	q
f	0	0	0	0	0	q	q
g	0	0	0	0	0	0	q

Table 2: Ω-valued order R

$$E(x,y) = R(x,y) \wedge R(y,x)$$

E	a	b	c	d	e	f	g
a	r	p	0	0	0	0	0
b	p	r	0	0	0	0	0
c	0	0	s	q	q	0	0
d	0	0	q	1	q	0	0
e	0	0	q	q	1	0	0
f	0	0	0	0	0	q	0
g	0	0	0	0	0	0	q

Table 1: Ω-valued equality E

R	a	b	c	d	e	f	g
a	r	r	0	0	r	0	0
b	p	r	0	0	r	0	0
c	0	0	s	q	s	q	q
d	r	r	s	1	1	q	q
e	0	0	q	q	1	q	q
f	0	0	0	0	0	q	q
g	0	0	0	0	0	0	q

Table 2: Ω-valued order R

$$E(x, y) = R(x, y) \wedge R(y, x)$$

(M, E, R) is an Ω-lattice as an ordered structure.

$$\mu = \begin{pmatrix} a & b & c & d & e & f & g \\ r & r & s & 1 & 1 & q & q \end{pmatrix}.$$

$$\mu = \begin{pmatrix} a & b & c & d & e & f & g \\ r & r & s & 1 & 1 & q & q \end{pmatrix}.$$

The cuts of μ and the cuts of E represented by partitions are:

$$\begin{array}{ll}
\mu_0 = M\ ; & E_0 = M^2; \\
\mu_p = \{a, b, c, d, e\}\ ; & E_p = \{\{a, b\}, \{c\}, \{d\}, \{e\}\}; \\
\mu_q = \{c, d, e, f, g\}\ ; & E_q = \{\{c, d, e\}, \{f\}, \{g\}\}; \\
\mu_r = \{a, b, d, e\}\ ; & E_r = \{\{a\}, \{b\}, \{d\}, \{e\}\}; \\
\mu_s = \{c, d, e\}\ ; & E_s = \{\{c\}, \{d\}, \{e\}\}; \\
\mu_t = \mu_1 = \{d, e\}; & E_t = E_1 = \{\{d\}, \{e\}\}.
\end{array}$$

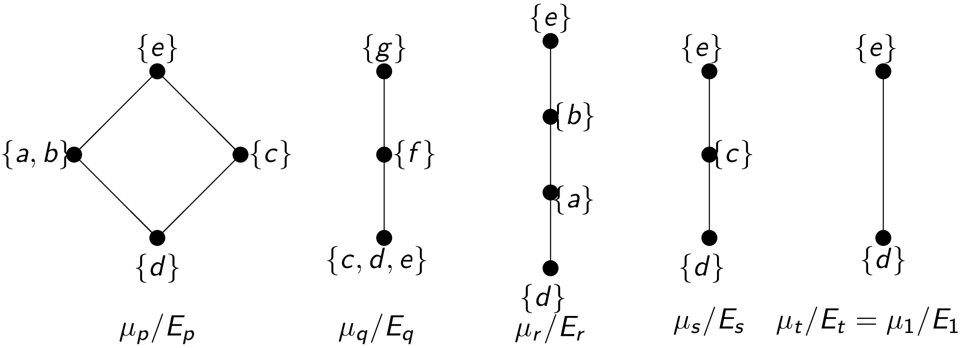


Figure 2: Quotient lattices

Two binary operations on M are constructed by means of pseudo-infima and pseudo-suprema. In this way, we obtain the bi-groupoid $\mathcal{M} = (M, \sqcap, \sqcup)$.

Two binary operations on M are constructed by means of pseudo-infima and pseudo-suprema. In this way, we obtain the bi-groupoid $\mathcal{M} = (M, \sqcap, \sqcup)$.

$\sqcap$	a	b	c	d	e	f	g
a	a	a	d	d	a	b^{**}	c^{**}
b	a	b	d	d	b	a^{**}	g^{**}
c	d	d	c	d	c	c^{*}	c^{*}
d	d	d	d	d	d	d^{*}	d^{*}
e	a	b	c	d	e	e^{*}	c^{*}
f	d^{**}	a^{**}	d^{*}	e^{*}	c^{*}	f	f
g	a^{**}	e^{**}	c^{*}	e^{*}	c^{*}	f	g

$\sqcup$	a	b	c	d	e	f	g
a	a	b	e	a	e	f^{**}	a^{**}
b	b	b	e	b	e	a^{**}	c^{**}
c	e	e	c	c	e	f	g
d	a	b	c	d	e	f	g
e	e	e	e	e	e	f	g
f	g^{**}	g^{**}	f	f	f	f	g
g	b^{**}	g^{**}	g	g	g	g	g

$\sqcup$	a	b	c	d	e	f	g
a	a	b	e	a	e	f^{**}	a^{**}
b	b	b	e	b	e	a^{**}	c^{**}
c	e	e	c	c	e	f	g
d	a	b	c	d	e	f	g
e	e	e	e	e	e	f	g
f	g^{**}	g^{**}	f	f	f	f	g
g	b^{**}	g^{**}	g	g	g	g	g

$(\mathcal{M}, E)$ is an Ω-lattice as an algebra.

References

References

- B. Šešelja, A. Tepavčević, Ω-algebras, 2015 IEEE Symposium Series on Computational Intelligence.
- B. Budimirović, V. Budimirović, B. Šešelja, A. Tepavčević, *E*-groups, Fuzzy Sets and Systems 289 (2016) 94-112.
- E.E. Eghosa, B. Šešelja, A. Tepavčević, *L*-*E*-lattices (submitted).
- E.E. Eghosa, B. Šešelja, A. Tepavčević, Complete Ω-lattices (submitted).
- O.S. Almabruk/Bleblou, B. Šešelja, A. Tepavčević, Normal Ω-Subgroups (submitted).

Thank you for your attention!