The omega-inequality problem for concatenation hierarchies of star-free languages

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- By a *positive* *variety of languages* we mean a correspondence $\mathcal{V}$ associating to each finite alphabet A a set $\mathcal{V}(A)$ of regular languages of A^* such that
  - $\mathcal{V}(A)$ is closed under finite union, finite intersection, and complementation;
  - if $\varphi : A^* \to B^*$ is a monoid homomorphism and $L \in \mathcal{V}(B)$ then $\varphi^{-1}(L) \in \mathcal{V}(A)$;
  - if L belongs to $\mathcal{V}(A)$ and $a \in A$, then so do the languages

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$$La^{-1} = \{w \in A^* : wa \in L\}.$$

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- *Boolean closure*: $\mathcal{V} \mapsto \mathcal{B}\,\mathcal{V}$, where

  $\mathcal{B}\,\mathcal{V}(A)$ is the closure of $\mathcal{V}(A)$ under finite union, finite intersection and complementation.

- *Polynomial closure*: $\mathcal{V} \mapsto \mathit{Pol}\,\mathcal{V}$, where

  $\mathit{Pol}\,\mathcal{V}(A)$ consists of all finite unions of languages of the form

  $$L_0 a_1 L_1 \cdots a_n L_n$$

  with $a_i \in A$ and $L_i \in \mathcal{V}(A)$.

- *Unambiguous polynomial closure*: $\mathcal{V} \mapsto \mathit{UPol}\,\mathcal{V}$, where

  $\mathit{UPol}\,\mathcal{V}(A)$ consists of all finite disjoint unions of *unambiguous* products of the form

  $$L_0 a_1 L_1 \cdots a_n L_n$$

  with $a_i \in A$ and $L_i \in \mathcal{V}(A)$.

We are interested in hierarchies constructed from a variety of languages $\mathcal{V}_0$ defined recursively by

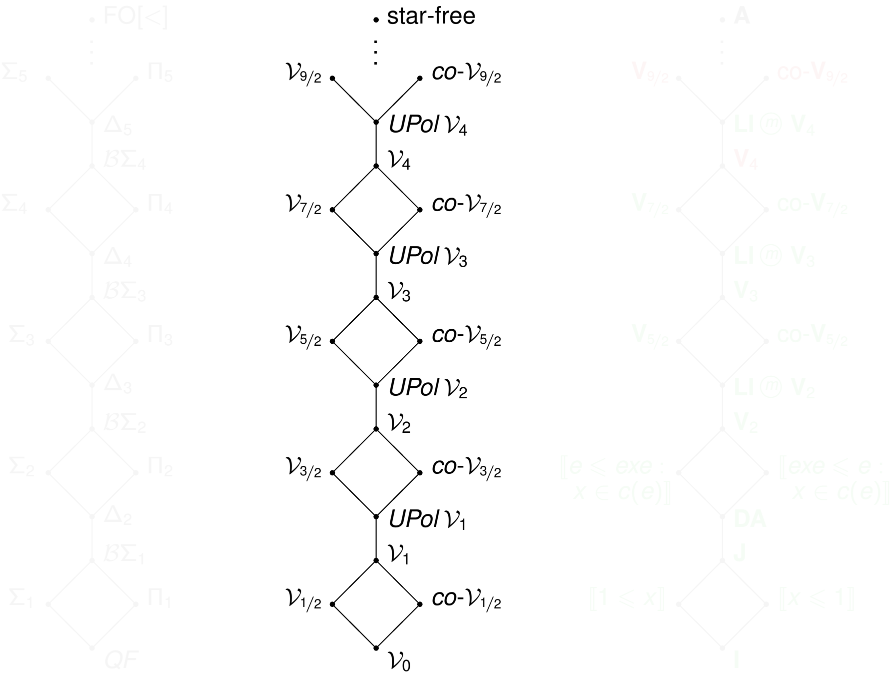
- $\mathcal{V}_{n+1/2} = Pol\,\mathcal{V}_n$,
- $\mathcal{V}_{n+1} = \mathcal{B}\,\mathcal{V}_{n+1/2}$.

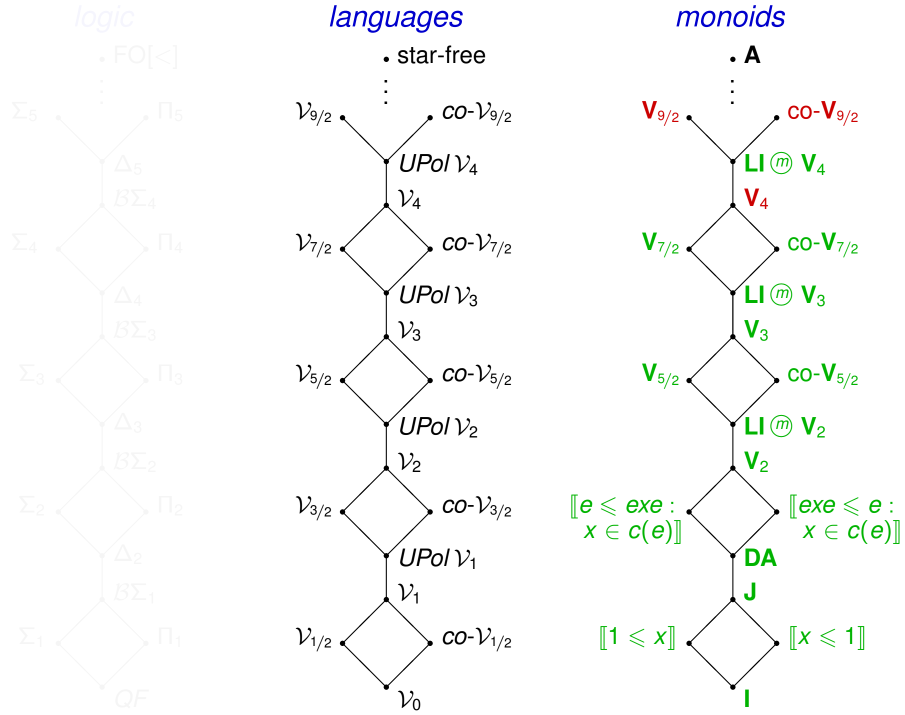
The half-levels of the hierarchy are positive varieties of languages, while the integer levels are varieties of languages.

The union of the hierarchy is the simultaneous *polynomial and Boolean closure* of $\mathcal{V}_0$.

In particular, starting from the *trivial variety of languages* ($\mathcal{V}_0(A) = \{\emptyset, A^*\}$), we obtain a hierarchy of star-free languages, which is known as the *Straubing-Thérien hierarchy*.

logic languages monoids





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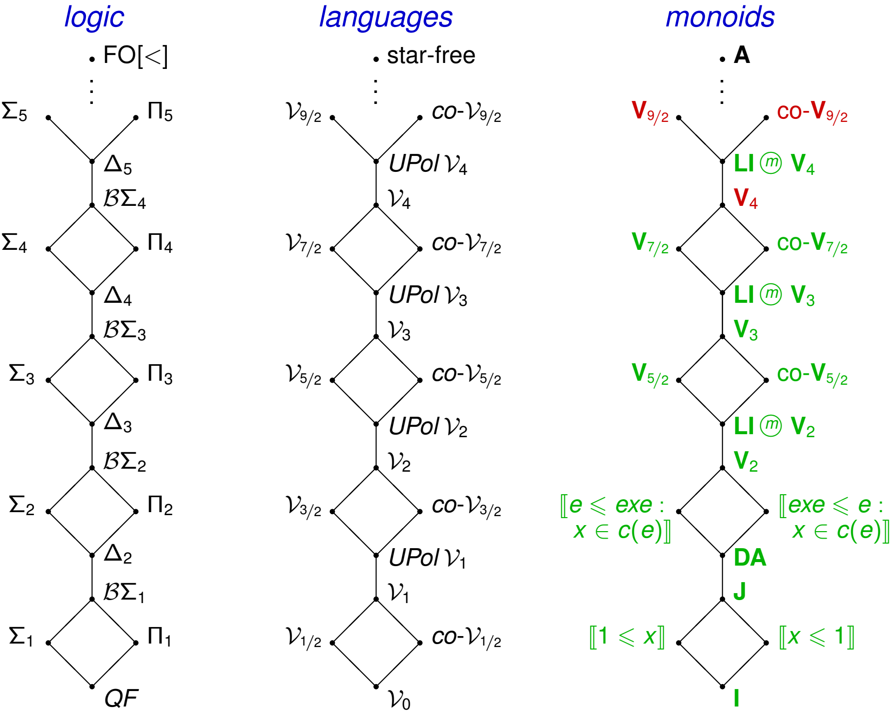

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A key technique has emerged in the proofs of decidability (of the membership problem) for the Straubing-Thérien hierarchy, which consists in solving the following harder problem for a pseudovariety **V** of ordered monoids:

- given a pair of elements (s, t) of a finite A-generated ordered monoid $(M, \leqslant)$, determine whether there exists a *witness* pair of elements (u, v) of the free profinite monoid $\overline{\Omega}_A \mathbf{M}$ such that

  (i) $u = s$ and $v = t$ in M;

  (ii) $u \leqslant v$ in $\mathbf{V}$;

Place and Zeitoun'2014 call this the *separation problem* because it is equivalent to deciding whether, given two regular languages $L, K \subseteq A^*$, there exists a **V**-recognizable language L' such that $L \subseteq L'$ and $L' \cap K = \emptyset$.

A standard compactness argument shows that, if **V** may be recursively enumerated, then so may be the negative instances of the separation problem.

Thus, for such **V**, to prove that the problem is decidable, it suffices to show that the positive instances may also be recursively enumerated.

Since $\overline{\Omega}_A\mathbf{M}$ is uncountable, it is better to reduce the set of witness pairs (u, v) that need to be considered, which leads to the following two properties of **V**, as suggested by work of JA-Steinberg'2000:

- *ω-reducibility (for $x \leqslant y$)*: if there is a witness (u, v) then there is one in which u and v are ω-words;
- *ω-inequality problem*: given two ω-words u and v, determine whether **V** satisfies $u \leqslant v$.

THEOREM

*If a recursively enumerable pseudovariety of ordered monoids **V** is ω-reducible (for $x \leqslant y$) and the ω-inequality problem for **V** is effectively solvable, then so is the separation problem for **V**.*

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Theorem

If a recursively enumerable pseudovariety of ordered monoids **V** *is ω-reducible (for $x \leqslant y$) and the ω-inequality problem for* **V** *is effectively solvable, then so is the separation problem for* **V**.

MAIN THEOREM

Let $\mathbf{V}$ *be a recursively enumerable pseudovariety of aperiodic monoids and let* $\mathbf{W} = \mathrm{Pol}\,\mathbf{V}$*. If the* ω*-inequality problem for* $\mathbf{W}$ *is decidable, then so is it for* $\mathrm{Pol}\,\mathrm{B}\,\mathbf{W}$*.*

COROLLARY

The ω*-inequality problem is decidable for all levels of the Straubing-Thérien hierarchy.*

- Let $\mathbf{W} = \operatorname{Pol}\mathbf{V}$.
- If an ω-inequality does not hold in $\operatorname{Pol}\mathrm{B}\,\mathbf{W}$, then it fails in some member of $\operatorname{Pol}\mathrm{B}\,\mathbf{W}$.
- Since $\operatorname{Pol}\mathrm{B}\,\mathbf{W}$ is recursively enumerable, it follows that the negative instances of the ω-inequality problem for $\operatorname{Pol}\mathrm{B}\,\mathbf{W}$ may be recursively enumerated.
- Thus, to prove decidability of the ω-inequality problem for $\operatorname{Pol}\mathrm{B}\,\mathbf{W}$, it remains to show that the positive instances may also be recursively enumerated.

- We show that the natural deductive calculus for ω-inequalities is complete, in the sense that an ω-inequality is valid in $\mathrm{Pol}\,\mathrm{B}\,\mathbf{W}$ if and only if it may be formally deduced from a certain recursively enumerable set of ω-inequalities valid in $\mathrm{Pol}\,\mathrm{B}\,\mathbf{W}$ which are given by a theorem of Pin of Weil'1997.
- The following are the main ingredients in this proof:

THEOREM (LIFTING FACTORIZATIONS)

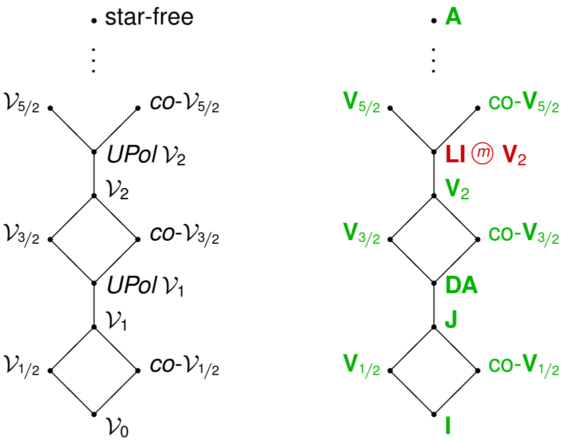
Let $\mathbf{W}$ be a polynomially closed pseudovariety of ordered monoids and let $u, v \in \overline{\Omega}_A\mathbf{M}$. If the inequality $u \leqslant v$ holds in $\mathbf{W}$ then, for every factorization $u = u_1 u_2$, there is a factorization $v = v_1 v_2$ such that each inequality $u_i \leqslant v_i$ holds in $\mathbf{W}$ $(i = 1, 2)$.

PROPOSITION (REPETITIONS)

Suppose, for each $i \geqslant 1$, v_i, x_i are ω-words in $\Omega^{\omega}_{A}\mathbf{A}$ such that $v_i = x_i v_{i+1}$. Then there exist indices i and j with $i < j$ such that $v_i = (x_i \cdots x_j)^{\omega} v_{j+1}$.

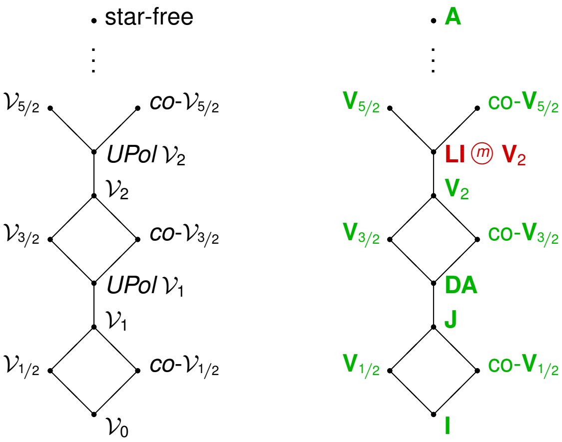
PROPOSITION (COMPLETENESS)

Let $\mathbf{V}$ be a pseudovariety of aperiodic monoids; $\mathbf{W} = \mathrm{Pol}\,\mathbf{V}$, and let Γ be the set of all trivial ω-inequalities together with all inequalities of the form $x^{\omega} \leqslant x^{\omega} y x^{\omega}$ such that the ω-inequality $y \leqslant x$ is valid in $\mathbf{W}$. If an ω-inequality $u \leqslant v$ is valid in $\mathrm{Pol}\,\mathrm{B}\,\mathbf{W}$, then $\Gamma \vdash u \leqslant v$.



Open problem

Let $\mathbf{V}$ *be a pseudovariety monoids. Does the operation* $\mathbf{V} \mapsto \mathbf{LI}$ ⓜ $\mathbf{V}$ *preserve decidability of the* ω*-identity problem?*



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Thank you!