SIGMA-COTORSION MODULES AND DEFINABILITY

JAN ŠŤOVÍČEK

Abstract. We prove that each Σ-cotorsion module over an associative ring with enough idempotents is contained in a definable class of cotorsion modules. This is closely related to the study of Σ-pure-injective objects in general finitely accessible additive categories and answers in affirmative a question posed by Guil Asensio and Herzog.

Contents

Introduction

Model theory has proved to be a powerful tool in theory of modules [19]. For a fixed ring R, we view modules as universal algebras with the additive group operations and multiplications by the elements of R. It has proved to be especially useful to consider axiomatizable classes of modules which are closed under direct sums and summands. Such classes are called definable and their relevance has been illustrated in several places in module theory and representation theory; see for example [4, 5]. A fundamental result by Ziegler [27], whose origins can be traced to Szmielew's solution [24] of the decidability problem for abelian groups, says that definable classes are completely determined by a so-called Ziegler spectrum. This is a topological space whose topology captures various model theoretic properties.

As noted by Crawley-Boevey [4], these techniques smoothly generalize to a much more general class of categories omnipresent in representation theory and geometry, namely to so-called finitely accessible additive categories. To

Date: November 25, 2011.
2000 *Mathematics Subject Classification.* 16D40 (primary), 03C60, 03C98, 16D50, 16E30 (secondary).
Key words and phrases. Sigma-cotorsion modules, model theory, finitely accessible additive categories, derived categories.
Research supported by the Eduard Čech Center for Algebra and Geometry (LC505).

be precise, Crawley-Boevey needs an extra assumption that the category in question admits all set-indexed products. Guil Asensio and Herzog tried in a series of papers [9, 10, 12, 13] to get rid of this extra assumption. For a nice overview we refer to [11]. As the authors mention, although straightforward analogs of several results indeed hold, these are far from easy modifications of Ziegler's or Crawley-Boevey's results. However, several problems remain unsolved as yet—for instance there seems to be no sensible definition of the topology on the analogue of the Ziegler spectrum.

In this paper, we establish properties of Σ-pure injective objects in finitely accessible additive categories. In fact, we do more. Using results from [4, 16], we know that each finitely accessible additive category is equivalent to Flat-R, the category of flat modules over a ring R with enough idempotents, and that pure injective objects precisely correspond to flat cotorsion modules. Studying Σ-pure injective objects then results to studying flat Σ-cotorsion modules. In fact, we have not been able to employ the flatness, so we study general Σ-cotorsion modules.

Our main results is that each Σ-cotorsion R-module is contained in a definable class in Mod-R consisting only of cotorsion modules. In particular, a pure submodule of a Σ-cotorsion module is Σ-cotorsion again, which answers in affirmative a question from [13]. However, note that similar to [11], our techniques are by no means similar to the classical case of Σ-pure injective modules. For instance, a Σ-cotorsion module need not be of countable character in the sense of [13]; see [2] for an example. Instead, we combine model and set-theoretic methods with modern homological algebra and theory of derived categories, which seems to be a new approach.

1. Preliminaries

1.1. **Rings with local units.** Let R be an associative ring. In order to get the connection with general finitely accessible categories later, we will not assume in general that R has a unit, but we will require existence of a complete orthogonal set of idempotents. That is, there will be a family $\{e_i \mid i \in I\}$ of idempotents in R such that

$$R = \bigoplus_{i \in I} e_i R = \bigoplus_{i \in I} R e_i.$$

All modules in this note will be unitary right R-modules, where a module M is unitary provided that $M = \bigoplus_{i \in I} M e_i$. Such modules have essentially the same homological properties as modules over a usual unitary ring. The only important difference in our case is that R as a module over itself might not be finitely generated in general, but it always decomposes to a direct sum of finitely generated (projective) summands. The category of all unitary right modules will be denoted by Mod-R, the full subcategory of finitely presented modules by mod-R. We will also use the category Flat-R of all flat modules, Proj-R of all projective modules and the category proj-R of all finitely generated projective modules.

It will prove useful soon to be able to identify module categories among general abelian categories. It is well-known and easy to see that a cocomplete abelian category $\mathcal{A}$ is equivalent to a module category for a ring with unit if and only if $\mathcal{A}$ has a small projective generator P. The word *small* means that $\mathrm{Hom}_{\mathcal{A}}(P,-) : \mathcal{A} \to \mathrm{Ab}$ commutes with coproducts. There is a completely analogous statement for rings with enough idempotents:

Lemma 1.1. *Let $\mathcal{A}$ be a cocomplete abelian category. Then $\mathcal{A}$ is equivalent to* Mod-R *for some ring R with enough idempotents if and only if there exists a set $\mathcal{P} \subseteq \mathcal{A}$ of small projective generators in $\mathcal{A}$.*

Proof. The "only if" part is clear. Assume now there is a set $\mathcal{P}$ of small projective generators in $\mathcal{A}$. Then we can take the ring

$$R = \left\{ f \in \mathrm{End}_{\mathcal{A}} \left(\bigoplus_{P \in \mathcal{P}} P \right) \mid (f \upharpoonright P) = 0 \text{ for all but finitely many } P \in \mathcal{P} \right\}.$$

One can then use standard arguments to check that the functor $F : \mathcal{A} \to$ Mod-R given by

$$F(X) = \bigoplus_{P \in \mathcal{P}} \mathrm{Hom}_{\mathcal{A}}(P, X)$$

on objects and obviously on morphisms, is a category equivalence. □

1.2. **Categories of complexes.** We will study several categories whose objects are complexes of R-modules. The interplay between these categories will be crucial, so we will explain the notation and terminology carefully. For the rest of the paper, we fix the following notation:

- $\mathbf{C}(\text{Mod-}R)$ stands for the abelian category of all chain complexes of R-modules.
- $\mathbf{C}^{\leq 0}(\text{Flat-}R)$ is the full subcategory of those complexes whose components are all flat and which vanish in positive degrees.
- $\mathbf{C}^{\leq 0}(\text{Proj-}R)$ stands, similarly, for the full subcategory of $\mathbf{C}(\text{Mod-}R)$ formed by complexes of projective modules concentrated only in non-positive degrees.
- $\mathbf{K}(\text{Mod-}R)$ denotes the homotopy category of complexes of R-modules. That is, the factor of $\mathbf{C}(\text{Mod-}R)$ modulo the ideal of all null-homotopic morphisms.
- $\mathbf{D}(\text{Mod-}R)$ stands for the derived category. That is, the localization of $\mathbf{K}(\text{Mod-}R)$ at the class of all quasi-isomorphisms.

First we focus on $\mathbf{C}(\text{Mod-}R)$. Using Lemma 1.1, we immediately see that this category has rather familiar properties.

Lemma 1.2. *Let R be a ring. Then the category $\mathbf{C}(\text{Mod-}R)$ is equivalent to* Mod-S *for some other ring S.*

Proof. In view of Lemma 1.1, we only have to construct a set of small projective generators for $\mathbf{C}(\text{Mod-}R)$. But one readily verifies that the complexes of the shape

$$\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow R \xrightarrow{1_R} R \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots,$$

shifted to all possible degrees, form such a set. □

Now make precise how to measure "size" of the complexes. To justify the terminology introduced below, we note that if one constructs the equivalence $F : \mathbf{C}(\text{Mod-}R) \to \text{Mod-}S$ as described in Lemmas 1.1 and 1.2, then X is a

κ-generated (or κ-presented) complex if and only if FX is a κ-generated (or κ-presented) S-module, respectively.

Definition 1.3. Let $X \in \mathbf{C}(\text{Mod-}R)$ and κ be an infinite cardinal number. The complex X is called κ-generated (κ-presented) if each component X^i is κ-generated (κ-presented, respectively) as an R-module.

Next we look at short exact sequences in $\mathbf{C}(\text{Mod-}R)$. Although $\mathbf{C}(\text{Mod-}R)$ has the natural exact structure because it is an abelian category, we find it often useful to use a different notion of short exact sequences. For a precise definition of what an exact structure is and their basic properties we refer to [18, App. A].

Focusing back on $\mathbf{C}(\text{Mod-}R)$, there is a well-known exact structure formed by semisplit short exact sequences. We remind that a short exact sequence $0 \to X \to Y \to Z \to 0$ of complexes is called *semisplit* if it splits in each component $0 \to X^i \to Y^i \to Z^i \to 0$. This exact structure is Frobenius, that is:

- The projective and injective objects with respect to the exact structure coincide. In our case these objects are precisely those complexes which are formed by splicing split exact sequences in Mod-R together. Such complexes X are called *contractible* and they are characterized by the property that X is sent to a zero object by the functor $\mathbf{C}(\text{Mod-}R) \to \mathbf{K}(\text{Mod-}R)$.
- There are enough projectives and enough injectives. In our case this means that for each $X \in \mathbf{C}(\text{Mod-}R)$ there is a semisplit monomorphism $X \to E(X)$ and a semisplit epimorphism $P(X) \to X$ such that $E(X)$ and $P(X)$ are contractible.

We recall that a morphism in $\mathbf{C}(\text{Mod-}R)$ is null-homotopic if and only if it factors through a projective object in the semisplit exact structure. Thus, the homotopy category $\mathbf{K}(\text{Mod-}R)$ can be considered as the stable category of $\mathbf{C}(\text{Mod-}R)$ modulo projectives. We refer to [14, §1] for details.

If $X, Y \in \mathbf{C}(\text{Mod-}R)$, we use the notation $\text{Ext}^i_R(X, Y)$ for the homomorphism group $\text{Hom}_{\mathbf{D}(\text{Mod-}R)}(X, Y[i])$. This consistently extends the definition of extension groups for modules, but one should not confuse these Ext's with usual Ext-groups in $\mathbf{C}(\text{Mod-}R)$ viewed as an abelian category. We will not use the latter Ext's at all, so there should be little risk of misunderstanding. If $\mathcal{C}$ is a class of complexes, we use the notation ${}^\perp\mathcal{C}$ for the class

$${}^\perp\mathcal{C} = \{X \in \mathbf{C}(\text{Mod-}R) \mid \text{Ext}^i_R(X, C) = 0 \text{ for each } C \in \mathcal{C} \text{ and } i \geq 0\}.$$

Finally, we recall important results regarding the construction of the derived category $\mathbf{D}(\text{Mod-}R)$.

Proposition 1.4. *The localization functor* $Q : \boldsymbol{K}(\text{Mod-}R) \to \boldsymbol{D}(\text{Mod-}R)$ *admits both a left adjoint*

$$\mathbf{p} : \boldsymbol{D}(\text{Mod-}R) \to \boldsymbol{K}(\text{Mod-}R)$$

and a right adjoint

$$\mathbf{i} : \boldsymbol{D}(\text{Mod-}R) \to \boldsymbol{K}(\text{Mod-}R).$$

More precisely, we have the following properties:

(1) *Both* $\mathbf{p}$ *and* $\mathbf{i}$ *are fully-faithful.*
(2) *The (co)units of adjunction* $\mathbf{p}X \to X$ *and* $X \to \mathbf{i}X$ *are quasi-isomorphisms for each* $X \in \boldsymbol{K}(\text{Mod-}R)$.
(3) *If* $P \in \operatorname{Im}\mathbf{p}$*, then the natural morphism*

$$\operatorname{Hom}_{\boldsymbol{K}(\text{Mod-}R)}(P, X) \longrightarrow \operatorname{Hom}_{\boldsymbol{D}(\text{Mod-}R)}(P, X)$$

induced by Q *is an isomorphism for any* $X \in \boldsymbol{K}(\text{Mod-}R)$. *Similarly, if* $I \in \operatorname{Im}\mathbf{i}$*, then*

$$\operatorname{Hom}_{\boldsymbol{K}(\text{Mod-}R)}(Y, I) \longrightarrow \operatorname{Hom}_{\boldsymbol{D}(\text{Mod-}R)}(Y, I)$$

is an isomorphism for each $Y \in \boldsymbol{K}(\text{Mod-}R)$.
(4) *Any bounded above complex of projectives is in* $\operatorname{Im}\mathbf{p}$. *Dually, any bounded below complex of injectives is in* $\operatorname{Im}\mathbf{i}$.

Proof. The statement was essentially proved by Spaltenstein [21] and later simplified by Bökstedt and Neeman [3]. □

The formulas in (3) of Proposition 1.4 show that the adjoints $\mathbf{p}$ and $\mathbf{i}$ are very important for computation, since working with homomorphisms in $\mathbf{K}(\text{Mod-}R)$ is usually much easier than direct computations with left or right fractions in $\mathbf{D}(\text{Mod-}R)$. Note also that statement (4) in the proposition in fact says that if $X \in \text{Mod-}R$, then $\mathbf{p}X$ is (up to isomorphism in $\mathbf{K}(\text{Mod-}R)$) any projective resolution of X and $\mathbf{i}X$ is any injective coresolution of X. Similarly, one can rather easily compute $\mathbf{p}X$ for a bounded above complex X and $\mathbf{i}Y$ for a bounded below complex Y, see [15, Lemma I.4.6]. Computing $\mathbf{p}X$ and $\mathbf{i}Y$ for general unbounded complexes is, however, somewhat more tricky; we refer to [21] and [3] for details.

Sometimes, one can compute homomorphism groups in $\mathbf{D}(\text{Mod-}R)$ directly, without evaluating $\mathbf{p}$. For this part, we will need the following simple result of relative homological algebra:

Lemma 1.5. *Let* $\mathcal{F}$ *be a class of modules and* $X \in \text{Mod-}R$ *such that* $\operatorname{Ext}^i_R(F, X) = 0$ *for each* $i \geq 1$. *Let* Y *be a bounded above complex with all terms in* $\mathcal{F}$. *Then the natural morphism*

$$\operatorname{Hom}_{\boldsymbol{K}(\text{Mod-}R)}(Y, X[i]) \longrightarrow \operatorname{Hom}_{\boldsymbol{D}(\text{Mod-}R)}(Y, X[i]) \quad (= \operatorname{Ext}^i_R(Y, X))$$

is an isomorphism for each $i \in \mathbb{Z}$.

Proof. By [15, Lemma I.4.6], we can find a quasi-isomorphism $q : \mathbf{p}Y \to Y$ such that $\mathbf{p}Y$ is a bounded above complex of projective modules. Let C be the mapping cone of q. Then C is a bounded above acyclic complex and one easily proves by induction that the images of $d^{j-1} : C^{j-1} \to C^j$, which we denote by Z^j, satisfy:

$$\operatorname{Ext}^i_R(Z^j, X) = 0 \text{ for each } j \in \mathbb{Z},\ i \geq 1 \text{ and } F \in \mathcal{F}.$$

Therefore, also the complex of abelian groups

$$\cdots \to \operatorname{Hom}_R(C^{j+1}, X) \to \operatorname{Hom}_R(C^j, X) \to \operatorname{Hom}_R(C^{j-1}, X) \to \dots$$

is acyclic. Now, the additive functor $\operatorname{Hom}_R(-, X) : \text{Mod-}R \to \text{Ab}$ extends componentwise to a contravariant triangulated functor $H_X : \mathbf{K}(\text{Mod-}R) \to$

$\mathbf{K}(\mathrm{Ab})$. When looking at the long exact sequence of homologies for the triangle

$$H_X C \longrightarrow H_X Y \stackrel{H_X(q)}{\longrightarrow} H_X(\mathbf{p}X) \longrightarrow (H_X C)[1]$$

in $\mathbf{K}(Ab)$, one immediately sees that q induces an isomorphism

$$\mathrm{Hom}_{\mathbf{K}(\text{Mod-}R)}(Y, X[i]) \longrightarrow \mathrm{Hom}_{\mathbf{K}(\text{Mod-}R)}(\mathbf{p}Y, X[i])$$

for each $i \in \mathbb{Z}$. This, together with Proposition 1.4, finishes the proof. □

2. On projective resolutions of flat modules

Let $M \in \text{Mod-}R$ and κ be a cardinal number. We call M to be *strongly κ-presented* if M has a projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

such that P_i is a κ-presented projective module for each $i \geq 0$. Normally, for a κ-presented module to be strongly κ-presented we need the base ring to be right coherent, but this might be to restrictive in the study of Σ-cotorsion modules. However, we will see that for flat modules the two concepts are always equivalent.

We start with a refinement of the well-known result by Lazard that every flat module is a direct limit of finitely generated projective modules:

Lemma 2.1. *Let κ be an infinite cardinal and F be a κ-presented flat module. Then there is a direct system $(P_i \mid i \in I)$ of finitely generated projective modules indexed by a set I of cardinality $\leq \kappa$ such that $F = \varinjlim P_i$.*

Proof. The proof is essentially contained in the proof of [8, Lemma 1.2.9]. We know from there that any homomorphism $f : M \to F$ with M finitely presented factors through a finitely generated projective module. On the other hand, $F = \varinjlim M_i$ for some direct system $(M_i \mid i \in I)$ of finitely presented modules such that $|I| \leq \kappa$. By the proof of $(c) \implies (a)$ of [8, Lemma 1.2.9], it is possible to construct a direct system $(P_i \mid i \in I)$ of finitely presented free modules with a different order on I such that $F = \varinjlim P_i$. □

Now we can prove the wanted coherency result:

Lemma 2.2. *Let κ be an infinite cardinal and F be a flat module. Then F is κ-presented if and only if it is strongly κ-presented.*

Proof. The "if" part is obvious. Let us prove the "only if" part. If F is κ-presented, we can express F as $\varinjlim P_i$ by Lemma 2.1 where $(P_i \mid i \in I)$ is a direct system of finitely generated projective modules and $|I| \leq \kappa$. Now, we can form the well-known exact sequence

$$\cdots \to \bigoplus_{i<j<k} P_i \to \bigoplus_{i<j} P_i \to \bigoplus_{i} P_i \to F \to 0$$

which is in fact a projective resolution of F consisting of κ-generated projective modules. Hence F is strongly κ-presented. □

3. Ext and direct limits

In general, the Ext-functor is not very well-behaved with respect to taking direct limits in its arguments. Under certain assumptions, however, things work well. In this section, we will generalize some of these results from the setting of modules to the setting of complexes.

3.1. **Pseudofiltrations and Eklof's Lemma.** We start with the so-called Eklof Lemma [8, 3.1.2]. In general, if $(M_i \mid i \in I)$ is a direct system of modules and $X \in \text{Mod-}R$ such that $\text{Ext}^1_R(M_i, X) = 0$ for each $i \in I$, one cannot conclude that also $\text{Ext}^1_R(\varinjlim M_i, X) = 0$. However, one can do so for well-ordered continuous systems with some condition on mapping cones as we are going to show. We need a definition first.

Definition 3.1. Let τ be an ordinal number and $(X_\alpha \mid \alpha \leq \tau)$ be a well-ordered direct system of complexes from $\mathbf{C}(\text{Mod-}R)$. Let, moreover, the system be continuous, that is, $X_\alpha = \varinjlim_{\beta<\alpha} X_\beta$ for each limit ordinal $\alpha \leq \tau$.

If $X \in \mathbf{C}(\text{Mod-}R)$, we say that $(X_\alpha \mid \alpha \leq \tau)$ is a *pseudofiltration* of X if $X_0 = 0$ and $X_\tau = X$. We call $(X_\alpha \mid \alpha \leq \tau)$ a *filtration* of X if in addition the morphisms $X_\alpha \to X_{\alpha+1}$ are monomorphisms for each $\alpha < \tau$. A filtration is called *semisplit* if all the morphisms $X_\alpha \to X_{\alpha+1}$ are semisplit.

In the sequel, we will be mostly interested in objects of $\mathbf{C}(\text{Mod-}R)$ only up to isomorphism in the derived category. In such a case, we can quite easily pass from a pseudofiltration to a semisplit filtration:

Lemma 3.2. *Let $X \in \mathbf{C}(\text{Mod-}R)$ and $(X_\alpha \mid \alpha \leq \tau)$ be a pseudofiltration of X. Then there is a semisplit filtration $(Y_\alpha \mid \alpha \leq \tau)$ of a complex $Y = Y_\tau$ and quasi-isomorphisms $q_\alpha : Y_\alpha \to X_\alpha$ for each $\alpha \leq \tau$ such that the following squares commute for each $\alpha \leq \beta \leq \tau$:*

$$\begin{array}{ccc} Y_\beta & \xrightarrow{q_\beta} & X_\beta \\ \uparrow & & \uparrow \\ Y_\alpha & \xrightarrow{q_\alpha} & X_\alpha \end{array}$$

Proof. The semisplit filtration and the quasi-isomorphisms $q_\alpha : Y_\alpha \to X_\alpha$ will be constructed by induction on α. We put $Y_0 = 0$ and $q_0 = 0$. For limit ordinals α, we will take $q_\alpha = \varinjlim_{\beta<\alpha} q_\beta$. The fact that q_α is a quasi-isomorphism follows from exactness of taking direct limits.

Let us assume that $\alpha = \beta + 1$ and $q_\beta : Y_\beta \to X_\beta$ has already been constructed. Denote by $f : X_\beta \to X_\alpha$ the corresponding map from the pseudofiltration of X, and by $e : Y_\beta \to E(Y_\beta)$ a semisplit monomorphism with $E(Y_\beta)$ contractible. Then we can form the commutative square:

$$\begin{array}{ccc} Y_\beta & \xrightarrow{q_\beta} & X_\beta \\ {\scriptstyle \binom{e}{fq_\beta}}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ E(Y_\beta) \oplus X_\alpha & \xrightarrow{(0,1)} & X_\alpha \end{array}$$

Clearly, the morphism on the left hand side is a semisplit monomorphism and the morphism at the bottom is a quasi-isomorphism. We just put

$Y_\alpha = E(Y_\beta) \oplus X_\alpha$ and take the morphism at the bottom for q_α. It is straightforward to check that all required squares commute. $\square$

Remark 3.3. In fact, it follows from the construction that all q_α are direct limits of split epimorphisms of complexes, but we do not have an application for this additional fact.

Now we can state the generalization of the Eklof Lemma from [8]:

Proposition 3.4. *Let $X, Y \in \boldsymbol{C}(\text{Mod-}R)$ and $(X_\alpha \mid \alpha \leq \tau)$ be a pseudofiltration of X. Denote for each $\alpha < \tau$ by C_α the mapping cone of $X_\alpha \to X_{\alpha+1}$. If $\operatorname{Ext}^1_R(C_\alpha, Y) = 0$ for each $\alpha < \tau$, then also $\operatorname{Ext}^1_R(X, Y) = 0$.*

Proof. By [3, §2] there is a complex $I = \mathbf{i}Y$, quasi-isomorphic to Y, such that

$$\operatorname{Hom}_{\mathbf{D}(\text{Mod-}R)}(Z, Y) \cong \operatorname{Hom}_{\mathbf{K}(\text{Mod-}R)}(Z, I)$$

for each $Z \in \mathbf{C}(\text{Mod-}R)$. Moreover, by Lemma 3.2, we can without loss of generality assume that $(X_\alpha \mid \alpha \leq \tau)$ is not just a pseudofiltration, but even a semisplit filtration. We will view all the semisplit monomorphisms as inclusions.

Under the assumptions just made we have an isomorphism $C_\alpha \cong X_{\alpha+1}/X_\alpha$ in $\mathbf{K}(\text{Mod-}R)$ and the hypothesis of the proposition translates to:

$$\operatorname{Hom}_{\mathbf{K}(\text{Mod-}R)}(X_{\alpha+1}/X_\alpha, J) = 0 \quad \text{for each } \alpha < \tau,$$

where put $J = I[1]$ for the sake of simplifying the notation. What we must prove is then that

$$\operatorname{Hom}_{\mathbf{K}(\text{Mod-}R)}(X, J) = 0.$$

Let, therefore, $c : X \to J$ be a chain complex morphism. We must prove that it is null-homotopic, that is, there is a collection of morphisms $(s^n \mid n \in \mathbb{Z})$, where $s^n : X^n \to J^{n-1}$, such that $c^n = ds^n + s^{n+1}d$ for each $n \in \mathbb{Z}$. We will construct by induction on α morphisms $s^n_\alpha : X^n_\alpha \to J^{n-1}$ such that

(1) $c^n \restriction X^n_\alpha = ds^n_\alpha + s^{n+1}_\alpha d$,
(2) $s^n_{\alpha+1} \restriction X_\alpha = s^n_\alpha$ for each $\alpha < \tau$.
(3) $s^n_\alpha = \varinjlim_{\beta<\alpha} s^n_\beta$ for each limit ordinal $\alpha \leq \tau$.

Then obviously can put $s^n = s^n_\tau$ to get the required null-homotopy. Regarding the induction, we put $s^n_0 = 0$ for each $n \in \mathbb{Z}$, and our choice of s^n_α on limit ordinals α is enforced by condition (3). Therefore, we only have to take care of ordinal successors.

Assume that $\alpha = \beta + 1$ and s^n_β have been constructed for all $n \in \mathbb{Z}$. Because the morphism $X_\beta \to X_\alpha$ is assumed to be a semisplit monomorphism, we can extend each $s^n_\beta : X_\beta \to J^{n-1}$ to a morphism $t^n_\alpha : X_\alpha \to J^{n-1}$. One readily checks that the morphisms

$$b^n_\alpha = dt^n_\alpha - t^{n+1}_\alpha d : X^n_\alpha \to J^n$$

define a chain complex morphism $b_\alpha : X_\alpha \to J$. However, it is likely to happen that $b_\alpha \neq c \restriction X_\alpha$. By the construction, nevertheless, the morphism

$$\Delta_\alpha = b_\alpha - (c \restriction X_\alpha) : X_\alpha \to J$$

vanishes on X_β, so we have an induced morphism $\overline{\Delta}_\alpha : X_\alpha/X_\beta \to J$, which is null-homotopic by our hypothesis. Let $(u_\alpha^n \mid n \in \mathbb{Z})$ be some null-homotopy, that is:

$$\overline{\Delta}_\alpha = du_\alpha^n - u_\alpha^{n+1} d \quad \text{for each } n \in \mathbb{Z}.$$

If we denote by $\pi_\alpha : X_\alpha \to X_\alpha/X_\beta$ the canonical semisplit epimorphism, we have:

$$c^n \restriction X_\alpha^n = b_\alpha^n - \Delta_\alpha^n = b_\alpha^n - \overline{\Delta}_\alpha^n \pi_\alpha^n = d(t_\alpha^n - u_\alpha^n \pi_\alpha^n) - (t_\alpha^{n+1} - u_\alpha^{n+1}\pi_\alpha^{n+1})d.$$

Hence we can put $s_\alpha^n = t_\alpha^n - u_\alpha^n \pi_\alpha^n$ and easily check that all required conditions are met. □

Remark 3.5. Let us shortly look at an interpretation of Proposition 3.4 in the case when all complexes X, X_α and Y are actually modules. If $(X_\alpha \mid \alpha \leq \tau)$ is a filtration, then $C_\alpha \cong X_{\alpha+1}/X_\alpha$ in $\mathbf{D}(\text{Mod-}R)$ for each $\alpha < \tau$. Hence the hypotheses translates to $\operatorname{Ext}_R^1(X_{\alpha+1}/X_\alpha, Y) = 0$ for each $\alpha < \tau$ and what we get is precisely [8, 3.2.1].

3.2. **A case when covariant Ext commutes with direct limits.** Let us recall that if κ is an infinite cardinal number, a direct system $(X_i \mid i \in I)$ is called *κ-direct* provided I is a κ-directed partial ordered set. That is, there is an upper bound in I for each subset $J \subseteq I$ of cardinality at most κ.

It is well-known that the canonical morphism $\varinjlim \operatorname{Hom}_R(Z, X_i) \to \operatorname{Hom}_R(Z, \varinjlim X_i)$ is an isomorphism provided that Z is a κ-presented module and $(X_i \mid i \in I)$ is a κ-direct system of modules. If, moreover, Z is strongly κ-presented, then also $\varinjlim \operatorname{Ext}_R^n(Z, X_i) \to \operatorname{Ext}_R^n(Z, \varinjlim X_i)$ are isomorphisms for each $n \geq 1$.

We aim to extend this statement for complexes, but we need a preparatory lemma first.

Lemma 3.6. *Let κ be an infinite cardinal and Z be a bounded above complex of strongly κ-presented modules. Then there exist a quasi-isomorphism $P \to Z$ in $\boldsymbol{C}(\text{Mod-}R)$ such that P is a bounded above complex of κ-generated projective modules.*

Proof. This follows immediately by combining the classical construction from [15, I.4.6] with the fact that the kernel of an epimorphism between two strongly κ-presented modules is again strongly κ-presented. □

Now we can state the proposition.

Proposition 3.7. *Let κ be an infinite cardinal, Z be a bounded above complex of strongly κ-presented modules, and $(X_i \mid i \in I)$ be a κ-direct system in $\boldsymbol{C}(\text{Mod-}R)$. Then the canonical morphism*

$$\varinjlim \operatorname{Ext}_R^n(Z, X_i) \longrightarrow \operatorname{Ext}_R^n(Z, \varinjlim X_i)$$

is an isomorphism for each $n \in \mathbb{Z}$.

Proof. We will prove the lemma only for $n = 0$, for other values of n it follows just by using the shifting automorphism of $\mathbf{C}(\text{Mod-}R)$. Let $P \to Z$ be a quasi-isomorphism as in Lemma 3.6. Then we have natural isomorphisms

$$\operatorname{Hom}_{\mathbf{D}(\text{Mod-}R)}(Z, -) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{D}(\text{Mod-}R)}(P, -) \xleftarrow{\sim} \operatorname{Hom}_{\mathbf{K}(\text{Mod-}R)}(P, -).$$

It is, therefore, enough to prove that the following canonical morphism is an isomorphism:

$$\varinjlim \operatorname{Hom}_{\mathbf{K}(\text{Mod-}R)}(P, X_i) \longrightarrow \operatorname{Hom}_{\mathbf{K}(\text{Mod-}R)}(P, \varinjlim X_i). \tag{i}$$

Denote $X = \varinjlim X_i$ and let $T = \operatorname{Tot}(\operatorname{Hom}(P, X))$ be the total Hom complex. That is, $T \in \mathbf{C}(\text{Ab})$ defined on components by

$$T^n = \prod_{j \in \mathbb{Z}} \operatorname{Hom}_R(P^j, X^{j+n}).$$

The differential $T^n \to T^{n+1}$ is defined by sending $(f_j \mid j \in \mathbb{Z}) \in T^n$, with $f_j : P^j \to X^{j+n}$, to

$$\big(f_{j+1}d + (-1)^{n+1} df_j \mid j \in \mathbb{Z}\big) \in T^{n+1}.$$

We refer to [26, 2.7.4] for details. It is well-known and easy to check that

$$H^0(T) = \operatorname{Hom}_{\mathbf{K}(\text{Mod-}R)}(P, Z).$$

Similarly, we define $T_i = \operatorname{Tot}(\operatorname{Hom}(P, X_i))$. The construction is functorial, so $(T_i \mid i \in I)$ is naturally a κ-direct system in $\mathbf{C}(\text{Ab})$. We are going to prove the proposition by proving a stronger statement that the natural morphism $\varinjlim T_i \to T$ is an isomorphism. Then the fact that (i) is an isomorphisms will follow just by looking at the homology groups of T and T_i in degree zero.

To prove that $\varinjlim T_i \to T$ is an isomorphism, we have to prove that the component morphisms of abelian groups $\varinjlim T_i^n \to T^n$ are isomorphisms for each $n \in \mathbb{Z}$. By the construction this amounts to prove that the following natural morphism is an isomorphism:

$$\varinjlim_{i \in I} \prod_{j \in \mathbb{Z}} \operatorname{Hom}_R(P^j, X_i^{j+n}) \longrightarrow \prod_{j \in \mathbb{Z}} \operatorname{Hom}_R(P^j, X^{j+n}).$$

This follows by combining the following two facts. First, the natural morphism

$$\varinjlim_{i \in I} \prod_{j \in \mathbb{Z}} \operatorname{Hom}_R(P^j, X_i^{j+n}) \longrightarrow \prod_{j \in \mathbb{Z}} \varinjlim_{i \in I} \operatorname{Hom}_R(P^j, X_i^{j+n})$$

is an isomorphism because I is $\aleph_0$-directed. Second, the natural morphisms

$$\varinjlim_{i \in I} \operatorname{Hom}_R(P^j, X_i^{j+n}) \to \operatorname{Hom}_R(P^j, X^{j+n})$$

are isomorphisms for each $j, n \in \mathbb{Z}$ because P^j are κ-generated projective modules and I is κ-directed. □

4. Concealed bounded complexes of flat modules

In the sequel an important role will be played by bounded complexes of flat modules. For technical reasons, however, we will usually need to consider different representatives in $\mathbf{D}(\text{Mod-}R)$ for such complexes which we will call concealed bounded. Before giving a definition, we motivate the concept by the following lemma:

Lemma 4.1. *Let $F \in \boldsymbol{C}^{\leq 0}(\text{Flat-}R)$:*

$$\cdots \xrightarrow{d^{-3}} F^{-2} \xrightarrow{d^{-2}} F^{-1} \xrightarrow{d^{-1}} F^{0} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

Then the following are equivalent:

(1) *There is a quasi-isomorphism $q : F \to G$ in $\boldsymbol{C}(\text{Mod-}R)$ such that $G \in \boldsymbol{C}^{\leq 0}(\text{Flat-}R)$ is a bounded complex.*
(2) *The sequences $0 \to \operatorname{Coker} d^{n-2} \to F^n \to \operatorname{Coker} d^{n-1} \to 0$ are pure exact for $n \ll 0$.*

Proof. (1) $\implies$ (2). Consider the mapping cone C_q of $q : F \to G$. It is an acyclic complex in $\mathbf{C}^{\leq 0}(\text{Flat-}R)$, so one easily proves by induction that the short sequences

$$0 \to \operatorname{Coker} d^{n-2} \to C_q^n \to \operatorname{Coker} d^{n-1} \to 0$$

are not only exact, but pure exact for any $n \leq 0$. This is because any short exact sequence with a flat last term is automatically pure.

Since G is bounded, we have $C_q^n = F^{n+1}$ for $n \ll 0$. Therefore,

$$0 \to \operatorname{Coker} d^{n-2} \to F^n \to \operatorname{Coker} d^{n-1} \to 0$$

is pure exact for $n \ll 0$.

(2) $\implies$ (1). Let $N \leq 0$ such that (2) holds for each $n \leq N$. Let G be the complex defined by

$$G : \qquad \cdots \to 0 \to \operatorname{Coker} d^{N-1} \to F^{N+1} \xrightarrow{d^{N+1}} \cdots \to F^{-1} \xrightarrow{d^{-1}} F^{0} \to 0 \to \cdots$$

Now, there is an obvious quasi-isomorphism $F \to G$, G is bounded, and $\operatorname{Coker} d^N$ is flat because it is a pure epimorphic image of F^N. $\square$

Definition 4.2. We call a complex $F \in \mathbf{C}^{\leq 0}(\text{Flat-}R)$ *concealed bounded* if it satisfies the equivalent conditions of Lemma 4.1.

The *essential width* of a concealed bounded complex F is defined to be the least $N \geq 0$ such that

$$0 \to \operatorname{Coker} d^{n-2} \to F^n \to \operatorname{Coker} d^{n-1} \to 0$$

is pure exact for each $n \leq -N + 1$.

A remark regarding the terminology: That a concealed bounded complex F has essential width N, means precisely that N is the smallest number such that there is a quasi-isomorphism $q : F \to G$ where G is a complex of flat modules concentrated in the N consecutive degrees $-N+1, -N+2, \ldots, 0$. The class of concealed bounded complexes has the favorable property of being closed under taking mapping cones and cokernels of semisplit monomorphisms:

Lemma 4.3. *Let $f : F \to F'$ be a chain complex homomorphism between concealed bounded complexes of flat modules of essential width at most N. Then:*

(1) *The mapping cone C_f of f is a concealed bounded complex of essential width at most $N + 1$.*
(2) *If f is a semisplit monomorphism, that* $\operatorname{Coker} f$ *is a concealed bounded complex of essential width at most $N + 1$.*

Proof. (1). There is an obvious quasi-isomorphism $q : F \to G$ where G is the following complex of flat modules of width at most N:

$$G: \qquad \cdots \to 0 \to \operatorname{Coker} d^{-N} \to F^{-N+2} \to \cdots \to F^{-1} \to F^0 \to 0 \to \cdots$$

There is also an analogous quasi-isomorphism $q' : F' \to G'$ and an obvious morphism $g : G \to G'$ making the following diagram commutative:

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "q"'] & F' \arrow[d, "q'"] \\ G \arrow[r, "g"] & G'. \end{tikzcd}$$

Therefore, there is a quasi-isomorphism $q'' : C_f \to C_g$ between the mapping cones. As C_g is a complex of flat modules concentrated in degrees $-N, -N+1, \dots, 0$, the same argument as for "(1) $\implies$ (2)" of Lemma 4.1 shows that the essential width of C_f is at most $N+1$.

(2). If f is a semisplit monomorphism, then clearly again $\operatorname{Coker} f \in \mathbf{C}^{\leq 0}(\text{Flat-}R)$. Moreover, there is a cokernel morphism $c : \operatorname{Coker} f \to C_f$ to the mapping cone C_f of f. Now, c is well-known to be a quasi-isomorphism, so also the composition $q'' \circ c : \operatorname{Coker} f \to C_g$ is a quasi-isomorphism. Here we use the notation as in the previous paragraph. Using once again the same argument as for "(1) $\implies$ (2)" of Lemma 4.1, we see that the essential width of $\operatorname{Coker} f$ is at most $N+1$. □

Now we will aim at the main result of this section. Roughly said, it states that given a concealed bounded complex whose all terms are projective, then the complex has many concealed bounded subcomplexes of projectives of the same essential width. For this purpose, we need two preparatory lemmas:

Lemma 4.4. *Let $P \in \mathbf{C}^{\leq 0}(\text{Proj-}R)$. Then there is a family $\mathcal{P}$ of subcomplexes of P with the following properties:*

1. *The inclusions $Q \subseteq P$ are semisplit for each $Q \in \mathcal{P}$.*
2. *$\mathcal{P}$ is closed under taking arbitrary sums and intersections.*
3. *For each infinite cardinal κ and each κ-generated subcomplex $X \subseteq P$, there is a κ-generated complex $Q \in \mathcal{P}$ containing X.*

Proof. By the classical result of Kaplansky [17], any projective module is a direct sum of countably generated projective modules. Let us fix such a decomposition for each component of P:

$$P^n = \bigoplus_{i \in I_n} P_i^n,$$

where I_n is some indexing set for each $n \leq 0$. Define

$$\mathcal{S} = \left\{ (J_n) \mid J_n \subseteq I_n \text{ and } d^{n-1}\left(\bigoplus_{i \in J_{n-1}} P_i^{n-1} \right) \subseteq \bigoplus_{i \in J_n} P_i^n \text{ for each } n \leq 0 \right\}.$$

Here, $d^{n-1} : P^{n-1} \to P^n$ is the differential in the complex P. Then we can define the family $\mathcal{P}$ of subcomplexes of P as:

$$\mathcal{P} = \left\{ Q \mid \exists (J_n) \in \mathcal{S} \text{ such that } Q^n = \bigoplus_{i \in J_n} P_i^n \text{ for each } n \leq 0 \right\}$$

It is easy to see that $\mathcal{P}$ satisfies properties (1) and (2). For (3), fix some infinite κ and a κ-generated subcomplex $X \subseteq P$. We define an element $(J_n) \in \mathcal{S}$ as follows. For each $n \in \mathbb{Z}$, we take $L_{n,0} \subseteq I_n$ smallest possible such that $X^n \subseteq \bigoplus_{i \in L_{n,0}} P_i^n$. Further, we inductively define for each $m \geq 1$ a subset $L_{n,m} \subseteq I_n$ as the smallest possible set containing $L_{n,m-1}$ and such that $d^{n-1}(\bigoplus_{i \in L_{n-1,m-1}} P_i^{n-1}) \subseteq \bigoplus_{i \in L_{n,m}} P_i^n$. Finally, we take $J_n = \bigcup_{m \geq 0} L_{n,m}$. It follows from the construction that $(J_n) \in \mathcal{S}$ and $|J_n| \leq \kappa$ for each $n \in \mathbb{Z}$. Hence, the element $Q \in \mathcal{P}$ corresponding to $(J_n) \in \mathcal{S}$ has all the required properties. $\square$

Lemma 4.5. *Let κ be an infinite cardinal, $n \leq m$ two integers, and F a bounded complex of flat modules concentrated in degrees $n, \dots, m$. Let $f : X \to F$ and $g : Y \to X$ be morphisms in $\boldsymbol{C}(\text{Mod-}R)$ such that X is κ-presented and Y is κ-generated. Then the following hold:*

(1) *There is a κ-presented complex $G \in \boldsymbol{C}(\text{Flat-}R)$ concentrated in degrees $n, \dots, m$ such that $f : X \to F$ factors through G.*
(2) *If $fg = 0$ in $\boldsymbol{C}(\text{Mod-}R)$, then the factorization $X \xrightarrow{f'} G \to F$ of f can be taken so that $f'g = 0$.*

Proof. Straightforward and tedious, using properties of direct limits and induction on the width of F. $\square$

Now we can state the aforementioned result which guarantees existence of a rich system of concealed bounded subcomplexes in any concealed bounded complex consisting of projective modules.

Proposition 4.6. *Let N be a natural number, and $P \in \boldsymbol{C}^{\leq 0}(\text{Proj-}R)$ be κ-generated and concealed bounded of essential width at most N in the sense of Definition 4.2. Then there is a family $\mathcal{Q}$ of subcomplexes of P with the following properties:*

(1) *The inclusions $Q \subseteq P$ are semisplit for each $Q \in \mathcal{Q}$.*
(2) *Each $Q \in \mathcal{Q}$ is concealed bounded of essential width at most N.*
(3) *$\mathcal{Q}$ is closed under taking unions of chains.*
(4) *For each infinite cardinal κ and each κ-generated subcomplex $X \subseteq P$, there is a κ-generated complex $Q \in \mathcal{Q}$ containing X.*

Proof. Let us fix a family $\mathcal{P}$ of subcomplexes of P with the properties given by Lemma 4.4, and put

$$\mathcal{Q} = \{Q \in \mathcal{P} \mid Q \text{ is concealed bounded of essential width at most } N\}.$$

The properties (1)–(3) are easily seen to be satisfied by $\mathcal{Q}$. For (3), one just has to inspect Definition 4.2 and take into account that a direct limit of pure exact sequences is again pure. In fact, this even shows that $\mathcal{Q}$ is closed under taking arbitrary directed unions, but we will not use this fact.

Hence, we focus on (4). Let κ be an infinite cardinal and X a κ-generated subcomplex of P. We shall fix a quasi-isomorphism $q : P \to G$ such that $G \in \mathbf{C}^{\leq 0}(\text{Flat-}R)$ is concentrated in degrees $-N+1, \dots, 0$. Such a quasi-isomorphism must exist since P is assumed to be concealed bounded of essential width at most N. In order to construct a suitable $Q \in \mathcal{Q}$, we first construct:

- a chain $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \dots$ of κ-generated subcomplexes of P, all contained in $\mathcal{P}$, such that $X \subseteq Y_0$, and
- a direct system $E_0 \xrightarrow{e_0} E_1 \xrightarrow{e_1} E_2 \xrightarrow{e_2} \dots$ of κ-presented complexes of flat modules which are concentrated in degrees $-N+1, \dots, 0$,

along with commutative diagrams of the following form for each $i \geq 0$:

$$\begin{tikzcd}
Y_i \arrow[r, "\subseteq"] \arrow[d] & Y_{i+1} \arrow[r, "\subseteq"] \arrow[d] & P \arrow[d, "q"] \\
E_i \arrow[r, "e_i"'] \arrow[ur, dotted] & E_{i+1} \arrow[r] & G
\end{tikzcd}$$

The latter diagram needs a little explanation. The solid arrows represent morphisms in $\mathbf{C}(\text{Mod-}R)$ which make both squares commutative. The dotted arrow represents a morphism which only exists in $\mathbf{D}(\text{Mod-}R)$ and which makes the two triangles commutative.

The construction is rather straightforward. Let Y_0 be any κ-generated complex from $\mathcal{P}$ containing X; use Lemma 4.4. Using Lemma 4.5(1) and the fact that Y_0 is in fact κ-presented, we obtain E_0 and morphisms in $\mathbf{C}(\text{Mod-}R)$ making the following square commutative:

$$\begin{tikzcd}
Y_0 \arrow[r, "\subseteq"] \arrow[d] & P \arrow[d, "q"] \\
E_0 \arrow[r] & G
\end{tikzcd}$$

Let now $i \geq 0$ and assume Y_i and E_i have been constructed. That is, we have a morphisms $E_i \to P$ in $\mathbf{D}(\text{Mod-}R)$ defined by the fraction:

$$\begin{tikzcd}
 & G & \\
E_i \arrow[ur] & & P \arrow[ul, "q"']
\end{tikzcd}$$

Let $\mathcal{P}'$ be the subset of $\mathcal{P}$ formed by the complexes which are κ-generated and contain Y_i. One easily sees that $\mathcal{P}'$ together with inclusions is a κ-direct system and $\bigcup \mathcal{P}' = X$. Since E_i is κ-presented, hence all components of E_i are strongly κ-presented by Lemma 2.2, we can use Proposition 3.7 to obtain the isomorphisms:

$$\varinjlim_{Y \in \mathcal{P}'} \operatorname{Hom}_{\mathbf{D}(\text{Mod-}R)}(E_i, Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{D}(\text{Mod-}R)}(E_i, P),$$

$$\varinjlim_{Y \in \mathcal{P}'} \operatorname{Hom}_{\mathbf{D}(\text{Mod-}R)}(Y_i, Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{D}(\text{Mod-}R)}(Y_i, P).$$

These isomorphism allow us to obtain the following commutative diagram in $\mathbf{D}(\text{Mod-}R)$ for some $Y' \in \mathcal{P}'$:

$$\begin{tikzcd}
Y_i \arrow[r, "\subseteq"] \arrow[d] & Y' \arrow[r, "\subseteq"] & P \arrow[d, "q"] \\
E_i \arrow[rr] \arrow[ur, dotted] & & G
\end{tikzcd}$$

Indeed, we use the first isomorphism to find a morphism $E_i \to Y'$ in $\mathbf{D}(\text{Mod-}R)$ making the right hand trapezoid commutative, and then the second isomorphism to make to lower triangle commutative by possibly choosing a bigger $Y' \in \mathcal{P}'$. The dotted arrow in the diagram represents, as before, a morphism from $\mathbf{D}(\text{Mod-}R)$ which may not be represented in $\mathbf{C}(\text{Mod-}R)$. Now, we just put $Y_{i+1} = Y'$.

The construction of E_{i+1} is similar. Using Lemma 4.5, we find a κ-presented complex $G' \in \mathbf{C}^{\leq 0}(\text{Flat-}R)$, concentrated in degrees $-N+1, \dots, 0$, together with some morphisms making the following diagram commutative in $\mathbf{C}(\text{Mod-}R)$:

$$\begin{tikzcd} Y_i \arrow[r, "\subseteq"] \arrow[d] & Y_{i+1} \arrow[r, "\subseteq"] \arrow[d] & P \arrow[d, "q"] \\ E_i \arrow[r] & G' \arrow[r] & G \end{tikzcd}$$

Next we fix a κ-direct system $(G_j \mid j \in J)$ in $\mathbf{C}^{\leq 0}(\text{Flat-}R)$ such that all the G_j are κ-presented and concentrated in degrees $-N+1, \dots, 0$, and $G = \varinjlim G_j$. Such a κ-direct system does always exist, this is closely related to Lemma 4.5. Then using the isomorphism

$$\varinjlim_{j \in J} \operatorname{Hom}_{\mathbf{D}(\text{Mod-}R)}(E_i, G_j) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{D}(\text{Mod-}R)}(E_i, G),$$

we can find $j \in J$ and morphisms making the following diagram commute in the same sense as before:

$$\begin{tikzcd} Y_i \arrow[r, "\subseteq"] \arrow[d] & Y_{i+1} \arrow[r, "\subseteq"] \arrow[d] & P \arrow[d, "q"] \\ E_i \arrow[r] \arrow[ur, dotted] & G_j \arrow[r] & G \end{tikzcd}$$

Now we just put $E_{i+1} = G_j$ and define the morphism $e_i : E_i \to E_{i+1}$ in the obvious way. This concludes the construction.

Having constructed the chain $(Y_i \mid i < \omega)$ and the direct system (E_i, e_i), we are almost done. Let $Q = \bigcup Y_i$ and $E = \varinjlim E_i$; this yields a commutative diagram:

$$\begin{tikzcd} Q \arrow[r, "\subseteq"] \arrow[d] & X \arrow[d, "q"'] \\ E \arrow[r] & G \end{tikzcd}$$

Let us inspect what happens on homologies. Because the m-th homology functor is well defined on $\mathbf{D}(\text{Mod-}R)$, we get the following commutative diagram for each $m \in \mathbb{Z}$:

$$\begin{tikzcd} H^m(Y_0) \arrow[r] \arrow[d] & H^m(Y_1) \arrow[r] \arrow[d] & H^m(Y_2) \arrow[r] \arrow[d] & H^m(Y_3) \arrow[r] \arrow[d] & \dots \\ H^m(E_0) \arrow[r] \arrow[ur] & H^m(E_1) \arrow[r] \arrow[ur] & H^m(E_2) \arrow[r] \arrow[ur] & H^m(E_3) \arrow[r] \arrow[ur] & \dots \end{tikzcd}$$

Since homology commutes with direct limits in $\mathbf{C}(\text{Mod-}R)$, it is clear the the morphism $H^m(Q) \to H^m(E)$ is an isomorphism for each $m \in \mathbb{Z}$. Hence

$Q \to E$ is a quasi-isomorphism, and Q is concealed bounded of essential width at most N. Moreover, $Q \in \mathcal{P}$, $X \subseteq Q$, and Q is κ-generated. In particular, $Q \in \mathcal{Q}$ and it has all the required properties. $\square$

Remark 4.7. Note that the main obstacle to overcome in the construction above is that we have a priori no information on how many generators or relations will the homologies of Y_i or E_i have. The number of generators is probably not bounded by κ in general if R is not a right coherent ring.

5. Deconstruction in the regular case

In this section, we start with the central idea—the deconstruction. It follows the framework described for instance in [6, 25], but there are some extra technical complications in our case. The setup is as follows:

We take a ring R and a class $\mathcal{D}$ of cotorsion modules which is closed under taking arbitrary direct sums. If C is a Σ-cotorsion module, we can take $\mathcal{D} = \operatorname{Add} C$. Given a flat module F, we know that $F \in {}^{\perp}\mathcal{D}$ and we would like to find a filtration $(F_\alpha \mid \alpha \leq \tau)$ such that each $F_{\alpha+1}/F_\alpha$ is countably presented and in ${}^{\perp}\mathcal{D}$. There are some well-known examples preventing simple minded approaches—for example it is not always possible to take a filtration of F so that all the factors $F_{\alpha+1}/F_\alpha$ would be countably presented and flat again.

What we will actually do is the following. If $F \in \text{Flat-}R$ and $P \to F$ is a quasi-isomorphism with $P \in \mathbf{C}^{\leq 0}(\text{Proj-}R)$, we will find a semisplit filtration $(P_\alpha \mid \alpha \leq \tau)$ of P such that all $P_{\alpha+1}/P_\alpha$ are countably generated, concealed bounded, and in ${}^{\perp}\mathcal{D}$. For technical reasons, we will have to prove the same also for any bounded complex $F \in \mathbf{C}^{\leq 0}(\text{Flat-}R)$.

The main strategy is to start with a coarser filtration and refine it by induction. As in [6, 25], we need to use very different techniques depending on whether the minimal number of generators and relations of F is a regular or a singular cardinal. In this section we discuss the regular case. We start with a preparatory lemma first:

Lemma 5.1. *Let R be a ring, C be a module, and $(F_i, f_{ij} \mid i \in I)$ be a direct system of modules such that $\operatorname{Ext}^1_R(F_i, C) = 0$ for each $i \in I$. Then the following are equivalent:*

1. $\operatorname{Ext}^1_R(\varinjlim F_i, C) = 0$.
2. *For each family $(g_{ij} \mid i < j)$ of morphism $g_{ij} : F_i \to C$ such that*
$$g_{ik} = g_{ij} + g_{jk} f_{ij} \qquad \text{for each } i, j, k \in I \text{ with } i < j < k,$$
there is a family $(g_i \mid i \in I)$ of morphisms $g_i : F_i \to C$ such that
$$g_i = g_{ij} + g_j f_{ij} \qquad \text{for each } i, j \in I \text{ with } i < j.$$

Proof. It is well known that there is a following exact sequence for $\varinjlim F_i$:

$$\dots \xrightarrow{\delta_2} \bigoplus_{i_0 < i_1 < i_2} F_{i_0 i_1 i_2} \xrightarrow{\delta_1} \bigoplus_{i_0 < i_1} F_{i_0 i_1} \xrightarrow{\delta_0} \bigoplus_{i_0} F_{i_0} \to \varinjlim F_i \to 0$$

where $F_{i_0 i_1 \dots i_n} = F_{i_0}$ for all $i_0 < i_1 < \dots < i_n$ in I and

$$\big(\delta_0 \restriction F_{ij}\big)(x) = \big(x, -f_{ij}(x)\big) \in F_i \times F_j$$
$$\big(\delta_1 \restriction F_{ijk}\big)(x) = \big(x, -x, -f_{ij}(x)\big) \in F_{ik} \times F_{ij} \times F_{jk}$$

If we apply the functor $\mathrm{Hom}_R(-, C)$ to that long exact sequence, we get in general a complex. It is easy to see that $\mathrm{Ext}^1_R(\varinjlim F_i, C) = 0$ if and only if this complex is exact at $\mathrm{Hom}_R(\bigoplus_{i_0<i_1} F_{i_0 i_1}, C)$. But we have:

$$\begin{aligned} \mathrm{Hom}_R(\delta^0, C)\big((g_i)_i\big) &= (g_i - g_j f_{ij})_{i<j} \\ \mathrm{Hom}_R(\delta^1, C)\big((g_{ij})_{i<j}\big) &= (g_{ik} - g_{ij} - g_{jk} f_{ij})_{i<j<k} \end{aligned}$$

Hence, the exactness condition translates precisely to condition (2) of the statement. $\square$

Now, the fundamental tool for the regular case is included in the following proposition, which generalizes [22, Theorem 8] and [7, Theorem XII.3.3]. In fact, if we only needed to find a semisplit filtration of a projective resolution of a κ-presented flat module, we would be essentially finished after this proposition. As mentioned, however, we also need to handle bounded complexes of flat modules, which requires some extra work.

Proposition 5.2. *Let R be a ring, κ an uncountable regular cardinal, and let $\mathcal{D}$ be a class of modules closed under arbitrary direct sums. Let F be a module and $(F_\alpha \mid \alpha \le \kappa)$ be a pseudofiltration of F such that all F_α are $<\kappa$-generated modules for $\alpha < \kappa$. Suppose that $\mathrm{Ext}^1_R(F_\alpha, \mathcal{D}) = 0$ for all $\alpha < \kappa$. Then the following conditions are equivalent:*

(1) $\mathrm{Ext}^1_R(F, \mathcal{D}) = 0$.
(2) *There is a closed unbounded subset $S \subseteq \kappa$ such that $\mathrm{Hom}_R(F_\beta, C) \to \mathrm{Hom}_R(F_\alpha, C)$ is surjective for each $C \in \mathcal{D}$ and for all $\alpha, \beta \in S$, $\alpha < \beta$.*

Remark 5.3. Note that condition (2) precisely means that for each $\alpha, \beta \in S$, $\alpha < \beta$, the mapping cone of $F_\alpha \to F_\beta$ is in ${}^\perp\mathcal{D}$; see Lemma 1.5.

Proof. Let us for each $\alpha \le \beta \le \kappa$ denote the pseudofiltration morphism $F_\alpha \to F_\beta$ by $f_{\alpha\beta}$.

(2) $\implies$ (1). We can w.l.o.g. assume that $\mathrm{Hom}_R(f_{\alpha\beta}, C)$ are surjective for all $C \in \mathcal{D}$ and $\alpha < \beta < \kappa$. It is not difficult to see that the condition (2) of Lemma 5.1 is always satisfied in that case, since we can construct the maps $g_\alpha : F_\alpha \to C$, $\alpha < \kappa$, by induction on α for any $C \in \mathcal{D}$. Hence $\mathrm{Ext}^1_R(F, \mathcal{D}) = 0$. Another way to prove of the implication is via Proposition 3.4, taking into account Remark 5.3.

(1) $\implies$ (2). Possibly by restricting ourselves to indices in some closed unbounded subset of κ, we can always assume that whenever $\mathrm{Hom}_R(f_{\alpha\beta}, C)$ is not surjective for some $\alpha < \beta$ and $C \in \mathcal{D}$, then already $\mathrm{Hom}_R(f_{\alpha,\alpha+1}, C)$ was not surjective.

Suppose now for contradiction that $\mathrm{Ext}^1_R(F, \mathcal{D}) = 0$, but the set

$$E = \{\alpha < \kappa \mid (\exists C \in \mathcal{D})\ \mathrm{Hom}_R(f_{\alpha,\alpha+1}, C) \text{ is not surjective}\}$$

is stationary in κ; that is, it intersects every closed unbounded subset of κ. Fix $C_\alpha \in \mathcal{D}$ and $h_\alpha \in \mathrm{Hom}_R(F_\alpha, C_\alpha)$ such that g_α does not factorize through $f_{\alpha,\alpha+1}$ for each $\alpha \in E$. Put $C_\alpha = 0$ and $h_\alpha = 0$ for $\alpha \in \kappa \setminus E$.

We will inductively construct homomorphisms $g_{\alpha\beta} : F_\alpha \to \bigoplus_{\mu<\beta} C_\mu$ for all $\alpha < \beta \le \kappa$ such that

(1) $g_{\alpha,\alpha+1}$ is the composition of h_α with the canonical inclusion $C_\alpha \to \bigoplus_{\mu<\alpha+1} C_\mu$, and
(2) $g_{\alpha\gamma} = g_{\alpha\beta} + g_{\beta\gamma} f_{\alpha\beta}$ for all $\alpha < \beta < \gamma \leq \kappa$.

For $\beta = 1$, we just put $g_{01} = h_0$. Suppose we have constructed $g_{\alpha\beta}$ for all $\alpha < \beta < \gamma$ for some $\gamma \leq \kappa$. If $\gamma = \delta + 1$ for some δ, put $g_{\delta\gamma} = h_\delta$ and $g_{\alpha\gamma} = g_{\alpha\delta} + h_\delta f_{\alpha\delta}$ for each $\alpha < \delta$. If γ is a limit ordinal, define $g_{\alpha\gamma}$ as the maps g_α given by Lemma 5.1, condition (2)—we use here the fact that $\bigoplus_{\mu<\gamma} C_\mu \in \mathcal{D}$. It is straightforward to check the the maps defined in this way satisfy the required conditions.

Next put:

$$S = \left\{ \lambda < \kappa \mid \operatorname{Im} g_{\alpha\kappa} \subseteq \bigoplus_{\mu<\lambda} C_\mu \text{ for each } \alpha < \lambda \right\}$$

It is easy to check that S is closed unbounded in κ. Hence, the set S' consisting of the limit ordinals in S is closed unbounded too, and there is some $\lambda \in S' \cap E$.

Denote by π the canonical projection $\bigoplus_{\mu<\kappa} C_\mu \to C_\lambda$. First we show that $\pi g_{\lambda\kappa} = 0$. Choose an arbitrary $x \in F_\lambda$. Since the $F_\lambda = \varinjlim_{\alpha<\lambda} F_\alpha$ by continuity of the direct system, there is $\alpha < \lambda$ and $y \in F_\alpha$ such that $x = f_{\alpha\lambda}(y)$. We have the equality:

$$\pi g_{\alpha\kappa}(y) = \pi g_{\alpha\lambda}(y) + \pi g_{\lambda\kappa} f_{\alpha\lambda}(y)$$

But $\pi g_{\alpha\kappa}(y) = 0$ since $\lambda \in S$ and $\pi g_{\alpha\lambda}(y) = 0$ by definition of $g_{\alpha\lambda}$. Hence $0 = \pi g_{\lambda\kappa} f_{\alpha\lambda}(y) = \pi g_{\lambda\kappa}(x)$. The claim follows since $x \in F_\lambda$ was arbitrary.

On the other hand, we know that $g_{\lambda\kappa} = g_{\lambda,\lambda+1} + g_{\lambda+1,\kappa} f_{\lambda,\lambda+1}$. Composing this with π, we get:

$$0 = \pi g_{\lambda\kappa} = \pi g_{\lambda,\lambda+1} + \pi g_{\lambda+1,\kappa} f_{\lambda,\lambda+1} = h_\lambda + \pi g_{\lambda+1,\kappa} f_{\lambda,\lambda+1}$$

But this implies that h_λ factorizes through $f_{\lambda,\lambda+1}$, a contradiction to the choice of h_λ for $\lambda \in E$.

Hence, E is not stationary. Therefore, we can choose a closed unbounded subset $S \subseteq \kappa$ such that $S \cap E = \emptyset$ and (2) follows. □

Before giving the main statement of the section, we some more auxiliary lemmas. First a homological lemma:

Lemma 5.4. *Let $\mathcal{D} \subseteq \boldsymbol{D}(\text{Mod-}R)$ be a class of objects and consider a commutative square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle v} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

in $\boldsymbol{D}(\text{Mod-}R)$. If X', Y' and the triangle completions of f and u all belong to ${}^\perp\mathcal{D}$, then also the triangle completion of f' belongs to ${}^\perp\mathcal{D}$.

Proof. Let us denote by Z and Z' the triangle completions of f and f', respectively. Then we have a diagram with triangles in rows:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle v} & & \uparrow{\scriptstyle w} & & {\scriptstyle u[1]}\uparrow \\ X' & \xrightarrow{f'} & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array} \tag{ii}$$

Denote further by X'' the triangle completion of u. Let us assume, as in the statement, that $X', X'', Y', Z \in {}^{\perp}\mathcal{D}$. We should prove that $Z' \in {}^{\perp}\mathcal{D}$. When inspecting the long exact sequences coming from applying the functors $\mathrm{Hom}_{\mathbf{D}(\mathrm{Mod}\text{-}R)}(-, C)$ to the lower triangle in diagram (ii), where C runs over all objects in $\mathcal{D}$, one immediately sees that $\mathrm{Ext}^i_R(Z', \mathcal{D}) = 0$ for each $i \geq 2$.

It remains to prove that, under our assumption, also $\mathrm{Ext}^1_R(Z', \mathcal{D}) = 0$. If we apply $\mathrm{Hom}_{\mathbf{D}(\mathrm{Mod}\text{-}R)}(-, C)$ with $C \in \mathcal{D}$ to the morphism of triangles (ii), we obtain the commutative diagram of abelian groups with exact rows:

$$\begin{array}{ccccccc} & & \mathrm{Ext}^1_R(Z, C) & \longleftarrow & \mathrm{Hom}(X, C) & \xleftarrow{f^*} & \mathrm{Hom}(Y, C) \\ & & \downarrow & & \downarrow{\scriptstyle u^*} & & \downarrow{\scriptstyle v^*} \\ \mathrm{Ext}^1_R(Y', C) & \longleftarrow & \mathrm{Ext}^1_R(Z', C) & \longleftarrow & \mathrm{Hom}(X', C) & \xleftarrow{(f')^*} & \mathrm{Hom}(Y', C). \end{array}$$

Now, the assumption that $X'', Z \in {}^{\perp}\mathcal{D}$ implies that both u^* and f^* are epimorphisms. Therefore, $(f')^*$ must be an epimorphism. Finally, the assumption that $Y' \in {}^{\perp}\mathcal{D}$ together with the surjectivity of $(f')^*$ yields $\mathrm{Ext}^1_R(Z', C) = 0$. Hence $Z' \in {}^{\perp}\mathcal{D}$. □

We also need a lemma on complexes of flat modules:

Lemma 5.5. *Let $\mathcal{D}$ be a class of cotorsion modules and $F \in \mathbf{C}^{\leq 0}(\mathrm{Flat}\text{-}R) \cap {}^{\perp}\mathcal{D}$:*

$$\cdots \xrightarrow{d^{-3}} F^{-2} \xrightarrow{d^{-2}} F^{-1} \xrightarrow{d^{-1}} F^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

Then $\mathrm{Ext}^1_R(\mathrm{Coker}\, d^i, \mathcal{D}) = 0$ for each $i \in \mathbb{Z}$.

Proof. The statement is trivial for $i \geq 0$. Assume we are given $i < 0$ and consider the monomorphism $f^i : \mathrm{Im}\, d^i \to F^{i+1}$. Using Lemma 1.5 and the fact that $\mathrm{Ext}^{-i}_R(F, \mathcal{D}) = 0$, one immediately sees that $\mathrm{Hom}_R(f^i, C)$ is an epimorphism for each $C \in \mathcal{D}$. If one applies $\mathrm{Hom}_R(-, C)$ on the short exact sequence

$$0 \longrightarrow \mathrm{Im}\, d^i \xrightarrow{f^i} F^{i+1} \longrightarrow \mathrm{Coker}\, d^i \longrightarrow 0$$

and takes into account that F^{i+1} is a flat module, one immediately sees that $\mathrm{Ext}^1_R(\mathrm{Coker}\, d^i, \mathcal{D}) = 0$. □

Now, we can prove the desired statement by induction on width of the complex in question. In fact, we can completely disregard the original bounded complex of flat modules F and consider only its projective resolution instead. Recall that if F is κ-presented for an infinite cardinal κ, then it has a κ-generated projective resolution by Lemmas 2.2 and 3.6.

Proposition 5.6. *Let κ be an uncountable regular cardinal, N a natural number, and $\mathcal{D}$ a class of cotorsion modules closed under arbitrary direct sums. Let $P \in \boldsymbol{C}^{\leq 0}(\text{Proj-}R)$ be κ-generated and concealed bounded of essential width at most N in the sense of Definition 4.2. If $P \in {}^{\perp}\mathcal{D}$, there is a semisplit filtration $(P_\alpha \mid \alpha \leq \kappa)$ of P such that for each $\alpha < \kappa$:*

1. *P_α is concealed bounded of essential with at most N,*
2. *P_α is less than κ-generated, and*
3. *$P_{\alpha+1}/P_\alpha \in {}^{\perp}\mathcal{D}$.*

Proof. It is not difficult to see using Proposition 4.6 that there is a semisplit filtration $(Q_\alpha \mid \alpha \leq \kappa)$ of P satisfying (1) and (2). We are going to prove by induction on N that, by possibly leaving out some terms from the filtration, we can obtain a filtration satisfying (3) as well. To start with, denote for each $\alpha \leq \kappa$ by G_α the complex:

$$G_\alpha: \qquad \cdots \to 0 \to \operatorname{Coker} d^{-N} \to Q_\alpha^{-N+2} \to \cdots \to Q_\alpha^{-1} \to Q_\alpha^0 \to 0 \to \cdots$$

By the assumptions, one immediately deduces that the obvious morphisms $q_\alpha : Q_\alpha \to G_\alpha$ are quasi-isomorphisms, $(G_\alpha \mid \alpha \leq \kappa)$ is a pseudofiltration in $\mathbf{C}^{\leq 0}(\text{Flat-}R)$, and the following squares with obvious morphisms commute for each $\alpha \leq \beta \leq \kappa$:

$$\begin{tikzcd} Q_\beta \arrow[r, "q_\beta"] & G_\beta \\ Q_\alpha \arrow[u] \arrow[r, "q_\beta"] & G_\alpha. \arrow[u] \end{tikzcd}$$

Next we proceed by induction on N. If $N = 1$, then all G_α are just flat modules. Using Proposition 5.2 we can assume, possibly by leaving out some terms both in $(G_\alpha \mid \alpha \leq \kappa)$ and $(Q_\alpha \mid \alpha \leq \kappa)$, that $(G_\alpha \mid \alpha \leq \kappa)$ is a pseudofiltration such that

$$\operatorname{Hom}_R(G_{\alpha+1}, C) \longrightarrow \operatorname{Hom}_R(G_\alpha, C)$$

is an epimorphism for each $\alpha < \kappa$ and $C \in \mathcal{D}$. This translates by Lemma 1.5 to the fact that the mapping cone of $G_\alpha \to G_{\alpha+1}$ belongs to ${}^{\perp}\mathcal{D}$. Since the mapping cone is isomorphic to $Q_{\alpha+1}/Q_\alpha$ in $\mathbf{D}(\text{Mod-}R)$, we conclude that also $Q_{\alpha+1}/Q_\alpha \in {}^{\perp}\mathcal{D}$. This gives property (3) and finishes the proof for $N = 1$.

Assume now that $N > 1$ and we have proved the proposition for complexes of essential width at most $N - 1$. Let H_α be for each $\alpha \leq \kappa$ the complex

$$H_\alpha: \qquad \cdots \to 0 \to G_\alpha^{-N+1} \to G_\alpha^{-N+2} \to \cdots \to G_\alpha^{-1} \to 0 \to 0 \to \cdots,$$

giving rise to commutative diagrams in $\mathbf{C}(\text{Mod-}R)$ whose rows become triangles in $\mathbf{K}(\text{Mod-}R)$:

$$\begin{tikzcd} H_\beta[-1] \arrow[r, "d_{G_\beta}^{-1}"] & G_\beta^0 \arrow[r] & G_\beta \arrow[r] & H_\beta \\ H_\alpha[-1] \arrow[u] \arrow[r, "d_{G_\alpha}^{-1}"] & G_\alpha^0 \arrow[u] \arrow[r] & G_\alpha \arrow[u] \arrow[r] & H_\alpha. \arrow[u] \end{tikzcd} \tag{iii}$$

Using Lemma 1.5 and the fact that $G_\kappa \in {}^\perp\mathcal{D}$ (this is since $G_\kappa \cong P$ in $\mathbf{D}(\text{Mod-}R)$), one easily sees that also $H_\kappa[-1] \in {}^\perp\mathcal{D}$. Using the inductive hypothesis and Proposition 3.4, we can assume that for each $\alpha \leq \beta \leq \kappa$:

- the mapping cone $\tilde{H}_{\alpha\beta}[-1]$ of $H_\alpha[-1] \to H_\beta[-1]$ is in ${}^\perp\mathcal{D}$;
- the mapping cone $\tilde{G}^0_{\alpha\beta}$ of $G^0_\alpha \to G^0_\beta$ is in ${}^\perp\mathcal{D}$.

The latter fact actually easily follows from the construction, since all the morphisms $G^0_\alpha \to G^0_\beta$ are split monomorphisms between projective modules.

Let now $\alpha < \kappa$. We have $H_\alpha[-1]$, G^0_α, G_κ, $\tilde{H}_{\alpha\kappa} \in {}^\perp\mathcal{D}$. To this end, note that $H_\alpha[-1] = \tilde{H}_{0\alpha}[-1]$, G^0_α is a projective module, and $G_\kappa \cong P$ in $\mathbf{D}(\text{Mod-}R)$. Hence, by first applying Lemma 5.4 on the square

$$\begin{array}{ccc} H_\kappa[-1] & \xrightarrow{d^{-1}_{G_\kappa}} & G^0_\kappa \\ \uparrow & & \uparrow \\ H_\alpha[-1] & \xrightarrow{d^{-1}_{G_\alpha}} & G^0_\alpha, \end{array}$$

and second by using Lemma 5.5, we deduce also that:

- both G_α and Q_α belong to ${}^\perp\mathcal{D}$ for each $\alpha \leq \kappa$;
- $\operatorname{Ext}^1(H^0(G_\alpha), \mathcal{D}) = \operatorname{Ext}^1(H^0(Q_\alpha), \mathcal{D}) = 0$.

It is a standard fact that for each $\alpha \leq \beta \leq \kappa$ we can form a triangle in $\mathbf{K}(\text{Mod-}R)$ (and so also in $\mathbf{D}(\text{Mod-}R)$) consisting of the mapping cones of the vertical morphisms from diagram (iii):

$$\tilde{H}_{\alpha\beta}[-1] \longrightarrow \tilde{G}^0_{\alpha\beta} \longrightarrow \tilde{G}_{\alpha\beta} \longrightarrow \tilde{H}_{\alpha\beta}.$$

Here, $\tilde{G}_{\alpha\beta}$ is the mapping cone of $G_\alpha \to G_\beta$, which is isomorphic to Q_β/Q_α in $\mathbf{D}(\text{Mod-}R)$. When inspecting the long exact sequences coming from applying the functors $\operatorname{Hom}_{\mathbf{D}(\text{Mod-}R)}(-, C)$ to the latter triangle, where C runs over all objects in $\mathcal{D}$, one immediately sees that also:

- $\operatorname{Ext}^i_R(\tilde{G}_{\alpha\beta}, \mathcal{D}) = 0 = \operatorname{Ext}^i_R(Q_\beta/Q_\alpha, \mathcal{D})$ for each $\alpha \leq \beta \leq \kappa$ and $i \geq 2$.

What is still left to prove, though, is that, possibly by leaving out some indices of the pseudofiltrations again, we can achieve vanishing of $\operatorname{Ext}^1_R(\tilde{G}_{\alpha\beta}, C)$ for each $\alpha \leq \beta \leq \kappa$ and $C \in \mathcal{D}$. Using the triangles

$$G_\alpha \longrightarrow G_\beta \longrightarrow \tilde{G}_{\alpha\beta} \longrightarrow G_\alpha[1],$$

in $\mathbf{K}(\text{Mod-}R)$ and Lemma 1.5, this is equivalent to say that all the morphisms

$$\operatorname{Hom}_{\mathbf{K}(\text{Mod-}R)}(G_\alpha, C) \longleftarrow \operatorname{Hom}_{\mathbf{K}(\text{Mod-}R)}(G_\beta, C)$$

are surjective whenever $C \in \mathcal{D}$ and $\alpha \leq \beta \leq \kappa$. Since G_α and G_β both vanish in all positive degrees, this is equivalent to say that

$$\operatorname{Hom}_R(H^0(G_\alpha), C) \longleftarrow \operatorname{Hom}_R(H^0(G_\beta), C) \tag{iv}$$

is surjective. Note also that $H^0(G_\alpha) = \operatorname{Coker} d^{-1}_{G_\alpha}$ is certainly $< \kappa$-generated (even $< \kappa$-presented) whenever $\alpha < \kappa$. Hence we can apply Proposition 5.2 to the pseudofiltration $(H^0(G_\alpha) \mid \alpha \leq \kappa)$ and find a closed unbounded subset of $S \subseteq \kappa$ such that (iv) is satisfied for each $\alpha \leq \beta$, $\alpha, \beta \in S$. When restricting the filtration $(Q_\alpha \mid \alpha \leq \kappa)$ only to the indices in $S \cup \{\kappa\}$, we get all the wanted properties (1)–(3) from the statement of the proposition. $\square$

6. Deconstruction completed via Singular Compactness

In this section, we finish the deconstruction, proving one of the main results of the paper:

Theorem 6.1. *Let R be a ring with enough idempotents, $\mathcal{D}$ be a class of cotorsion R-modules which is closed under arbitrary direct sums, and let $P \in \boldsymbol{C}^{\leq 0}(\text{Proj-}R) \cap {}^{\perp}\mathcal{D}$ be a concealed bounded complex in the sense of Definition 4.2. Then there is a semisplit filtration $(P_\alpha \mid \alpha \leq \tau)$ of P such that for each $\alpha < \tau$:*

1. *P_α is concealed bounded,*
2. *P_α is countably generated, and*
3. *$P_{\alpha+1}/P_\alpha \in {}^{\perp}\mathcal{D}$.*

Remark 6.2. The theorem in particular applies to the case we are especially interested in—when P is a projective resolution of an arbitrary flat module.

Before proving the theorem, we need some preparation. The main obstacle unsolved in the last section is the case when the minimal number of generators for P is a singular cardinal. For this, we will use the so called Shelah's Singular Compactness Theorem. In this paper, we will give only the necessary facts about the theorem following the presentation in [6], because a full discussion would need a paper of its own. We refer to [6] or [7, XII.1.14 and IV.3.7] for a more comprehensive treatment.

The main idea, originally due to Shelah, is generalizing the usual concept of free modules over a unital ring. Given a ring S and an infinite cardinal μ, we may designate a class $\mathcal{F} \subseteq \text{Mod-}S$, satisfying certain conditions, whose members we call *$\mathcal{F}$-free* or, if there is no risk of confusion, just *"free"* modules. The precise and rather technical conditions on $\mathcal{F}$, involving the parameter μ, are discussed in detail in [6, §1.2], and the role of μ will become clear a little later.

We can now discuss the most important example for us of such generalized classes of "free" modules. In fact, we are interested in "free" complexes, but this is a negligible difference in view of Lemma 1.2. If we are to talk about such freeness in the category of complexes, we will always mean the corresponding concept in a fixed equivalent category of modules. First, however, we need a definition:

Definition 6.3. Let R be a ring and $\mathcal{S} \subseteq \mathbf{C}(\text{Mod-}R)$ be a class of complexes. Then a complex $X \in \mathbf{C}(\text{Mod-}R)$ is called *$\mathcal{S}$-filtered* if it possesses a filtration $(X_\alpha \mid \alpha \leq \tau)$ such that $X_{\alpha+1}/X_\alpha$ is isomorphic in $\mathbf{C}(\text{Mod-}R)$ to an element of $\mathcal{S}$ for each $\alpha < \tau$.

Lemma 6.4. *Let R be a ring, μ be an infinite cardinal, and $\mathcal{S} \subseteq \boldsymbol{C}(\text{Mod-}R)$ be a set of μ-presented complexes. Then the class $\mathcal{F}$ consisting of all $\mathcal{S}$-filtered complexes satisfies the conditions for generalized freeness for the cardinal μ.*

Proof. Let us fix the category equivalence $F : \mathbf{C}(\text{Mod-}R) \to \text{Mod-}S$ given by Lemma 1.2. This equivalence, as noted before Definition 1.3, sends μ-presented complexes to μ-presented S-modules. But the class $F(\mathcal{F}) \subseteq \text{Mod-}S$ of all (up to isomorphism) $F(\mathcal{S})$-filtered S-modules satisfies the condition for almost freeness for the cardinal μ by [6, §2, part III]. The proof in

[6] assumes that the ring S has a unit, but it reads unchanged also for rings with enough idempotents. $\square$

Another nice property of $\mathcal{S}$-filtered complexes, in fact closely related to the concept of freeness, is the so called Hill Lemma, which we will need later. The simplified version presented here is based on [23], though for semisplit $\mathcal{S}$-filtrations of X, which we will actually need the lemma only for, the proof would become somewhat easier:

Lemma 6.5. *Let R be ring, μ be an infinite cardinal, and $\mathcal{S} \subseteq \boldsymbol{C}(\mathrm{Mod}\text{-}R)$ a set of μ-presented complexes. Let $X \in \boldsymbol{C}(\mathrm{Mod}\text{-}R)$ be an $\mathcal{S}$-filtered complex. Then there is a family $\mathcal{H}$ of subcomplexes of X such that*

(1) *$0, X \in \mathcal{H}$.*
(2) *$\mathcal{H}$ is closed under taking arbitrary unions and intersections.*
(3) *For each $Z, Z' \in \mathcal{H}$ such that $Z \subseteq Z'$, the factor-complex Z'/Z is $\mathcal{S}$-filtered.*
(4) *For each $\kappa \geq \mu$ and a κ-generated subcomplex $Y \subseteq X$, there is a κ-presented complex $Z \in \mathcal{H}$ containing Y.*

Now we shall concentrate on the singular compactness theorem. It roughly says that if X is a module with a singular number of generators, and X has enough "free" submodules, then X is necessarily "free" itself. Let us now state the theorem precisely, again in the language of complexes:

Definition 6.6. Let R be a ring and $\mathcal{F} \subseteq \mathbf{C}(\mathrm{Mod}\text{-}R)$ be a class satisfying the conditions for generalized freeness (for some cardinal μ). For an uncountable regular cardinal κ, a complex $X \in \mathbf{C}(\mathrm{Mod}\text{-}R)$ is defined to be *κ-$\mathcal{F}$-free*, or simply *κ-"free"*, if there is a set $\mathcal{C}$ of less than κ-generated subcomplexes of X such that:

(1) every element of $\mathcal{C}$ is "free";
(2) every less than κ-generated subcomplex of X is contained in an element of $\mathcal{C}$; and
(3) $\mathcal{C}$ is closed under unions of well-ordered chains of length less than κ.

Proposition 6.7 (Shelah's Singular Compactness Theorem)**.** *Let R be a ring, μ an infinite cardinal, and $\mathcal{F} \subseteq \boldsymbol{C}(\mathrm{Mod}\text{-}R)$ be a class satisfying the conditions for generalized freeness for μ. Let $\lambda > \mu$ be a singular cardinal and X be a λ-generated complex such that X is κ-$\mathcal{F}$-free for all regular cardinals κ such that $\mu < \kappa < \lambda$. Them X is $\mathcal{F}$-free.*

Proof. In view of the translation between complexes and modules given by Lemma 1.2, the statement precisely corresponds to [6, 1.4]. $\square$

We are ready to prove Theorem 6.1 at this point:

Proof of Theorem 6.1. Suppose $\mathcal{D}$ is a class of cotorsion modules which is closed under arbitrary direct sums, and let $P \in \mathbf{C}^{\leq 0}(\mathrm{Proj}\text{-}R) \cap {}^{\perp}\mathcal{D}$ be a concealed bounded complex. If P is countably generated, we are done. Assume therefore that P is not countably generated and λ is the least cardinal such that P is λ-generated. We prove the theorem by induction on λ.

If λ is regular, we take a filtration $(P_\alpha \mid \alpha \leq \kappa)$ given by Proposition 5.6 for $\kappa = \lambda$. Now, each factor $P_{\alpha+1}/P_\alpha$ is less than λ-generated, belongs to

$\mathbf{C}^{\leq 0}(\text{Proj-}R) \cap {}^{\perp}\mathcal{D}$, and it is concealed bounded by Lemma 4.3. Hence, using the inductive hypothesis, we can refine the filtration to get one meeting the requirements of the conclusion of Theorem 6.1.

Let λ be singular. In this case, we denote by $\mathcal{S}$ a representative set of countably generated concealed bounded complexes from $\mathbf{C}^{\leq 0}(\text{Proj-}R) \cap {}^{\perp}\mathcal{D}$. Let $\mathcal{F}$ be the class of all $\mathcal{S}$-filtered complexes. Note that any such filtration is necessarily semisplit, since the components of complexes from $\mathcal{S}$ are projective. By Lemma 6.4, we can view the complexes in $\mathcal{F}$ as "free" for $\mu = \aleph_0$.

With this notation, what we must show is precisely that P is "free". By Proposition 6.7, this will reduce to showing that P is κ-"free" for each uncountable regular cardinal κ such that $\kappa < \lambda$.

We will show that P is "free" by induction on the essential width N of the complex P. First assume $N \leq 1$ and let κ be as required, that is κ is regular and $\aleph_0 < \kappa < \lambda$. Then P is in fact a projective resolution of a flat module. Let $\mathcal{Q}$ be a family of subcomplexes of P given by Proposition 4.6, and take

$$\mathcal{C} = \{Q \in \mathcal{Q} \mid Q \text{ is less than } \kappa\text{-generated}\}.$$

Since each $Q \in \mathcal{C}$ is again a projective resolution of a flat module, we automatically have $Q \in {}^{\perp}\mathcal{D}$. Hence, by the inductive hypothesis, each Q is $\mathcal{S}$-filtered, or in other words "free". Therefore, $\mathcal{C}$ has the properties required by Definition 6.6, showing that P is κ-free. Since this works for all regular $\aleph_0 < \kappa < \lambda$, P is "free".

Let now $N > 1$ and denote by H the truncated complex:

$$H: \qquad \cdots \longrightarrow P^{-3} \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots,$$

Then we have the following triangle in $\mathbf{K}(\text{Mod-}R)$:

$$H[-1] \xrightarrow{d_P^{-1}} P^0 \longrightarrow P \longrightarrow H.$$

Clearly, $H[-1]$ is λ-generated, has essential width at most $N-1$, and belongs to $\mathbf{C}^{\leq 0}(\text{Proj-}R) \cap {}^{\perp}\mathcal{D}$. We refer to Lemma 1.5 for the latter. In particular, $H[-1]$ is "free", or $\mathcal{S}$-filtered, by the inductive hypothesis. Fix a family $\mathcal{H}$ of subcomplexes of $H[-1]$ given by Lemma 6.5 (the Hill Lemma), and a family $\mathcal{Q}$ of subcomplexes of $H[-1]$ of essential width at most $N-1$ given by Proposition 4.6. We claim that the intersection $\mathcal{H} \cap \mathcal{Q}$ has the following properties:

(1) $\mathcal{H} \cap \mathcal{Q}$ is closed under taking unions of chains.
(2) For each infinite cardinal κ and each κ-generated subcomplex $Y \subseteq H[-1]$, there is a κ-generated complex $Z \in \mathcal{H} \cap \mathcal{Q}$ containing Y.

Indeed, (1) is clear from the properties of $\mathcal{H}$ and $\mathcal{Q}$, and for (2), given a κ-generated $Y \subseteq H[-1]$, we inductively construct a chain $Y \subseteq Z_0 \subseteq Q_0 \subseteq Z_1 \subseteq Q_1 \subseteq \dots$ of κ-generated complexes such that $Z_i \in \mathcal{H}$ and $Q_i \in \mathcal{Q}$ for each $i \geq 0$. Then clearly $Z = \bigcup Z_i = \bigcup Q_i \in \mathcal{H} \cap \mathcal{Q}$ is a κ-generated complex containing Y.

Having this notation, we define a set $\mathcal{C}'$ of subcomplexes of P as follows. We fix a decomposition $P^0 = \bigoplus_{i \in I} P_i^0$ of the projective module P^0 into countably generated summands. Then a subcomplex $C \subseteq P$ belongs to $\mathcal{C}'$

if it is of the form

$$C: \qquad \cdots \longrightarrow Z^{-2} \longrightarrow Z^{-1} \longrightarrow Z^0 \overset{d_P^{-1}\restriction Z^0}{\longrightarrow} \bigoplus_{i\in J} P_i^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots,$$

where the direct sum $\bigoplus_{i\in J} P_i^0$ is located in degree zero, and:

(1) The complex

$$Z: \qquad \cdots \longrightarrow Z^{-2} \longrightarrow Z^{-1} \longrightarrow Z^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots,$$

with Z^0 in degree zero, belongs to $\mathcal{H} \cap \mathcal{Q}$ (so in particular Z is a subcomplex of $H[-1]$); and

(2) $J \subseteq I$ such that $\mathrm{Im}(d_P^{-1} \restriction Z^0) \subseteq \bigoplus_{i\in J} P_i^0$;

Given such C, we have the following commutative diagram in $\mathbf{C}(\text{Mod-}R)$ with semisplit monomorphisms in columns and whose rows become triangles in $\mathbf{K}(\text{Mod-}R)$:

$$\begin{array}{ccccccc}
H[-1] & \xrightarrow{d_P^{-1}} & P^0 & \longrightarrow & P & \longrightarrow & H \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
Z & \longrightarrow & \bigoplus_{i\in J} P_i^0 & \longrightarrow & C & \longrightarrow & Z[1].
\end{array}$$

By the construction and Proposition 3.4, we now have Z, $\bigoplus_{i\in J} P_i^0$, P, $H[-1]/Z \in {}^{\perp}\mathcal{D}$. By Lemma 5.4, it follows that $C \in {}^{\perp}\mathcal{D}$. To summarize also other properties which follow from the construction in a straightforward manner, we have:

(1) Each $C \in \mathcal{C}$ is concealed bounded of essential width at most N and belongs to $\mathbf{C}^{\leq 0}(\text{Proj-}R) \cap {}^{\perp}\mathcal{D}$.
(2) For each infinite cardinal κ and each κ-generated subcomplex $Y \subseteq P$, there is a κ-generated complex $C \in \mathcal{C}'$ containing Y.
(3) $\mathcal{C}'$ is closed under taking unions of chains.

Finally, if κ is an uncountable regular cardinal such that $\kappa < \lambda$ and we take

$$\mathcal{C} = \{C \in \mathcal{C}' \mid C \text{ is less than } \kappa\text{-generated}\},$$

then $\mathcal{C}$ clearly corresponds to Definition 6.6, showing that P is κ-free. Then by Proposition 6.7 it follows that P is "free". This finishes the inner induction on N, and also the outer induction on λ. □

7. Definability

Here, we prove our main result, which gives a link from Σ-cotorsion modules to the first order theories of modules, and answers a question posed by Guil Asensio and Herzog:

Theorem 7.1. *Let R be a ring with enough idempotents and $\mathcal{D}$ be a class of cotorsion R-modules which is closed under taking arbitrary direct sums. Then there is a class $\overline{\mathcal{D}} \subset \text{Mod-}R$ such that $\mathcal{D} \subseteq \overline{\mathcal{D}}$, $\overline{\mathcal{D}}$ is definable and $\overline{\mathcal{D}}$ consists only of cotorsion modules.*

The immediate corollary is then:

Corollary 7.2. *Let R be a ring with enough idempotents and C be a Σ-cotorsion module. Then any pure submodule of C, any product of copies of C, as well as any module elementarily equivalent to C is again Σ-cotorsion. In particular, if the cotorsion envelope $CE(X)$ of an R-module X is Σ-cotorsion, then so is X itself.*

To prove the theorem, we again need some preparation. In this case, we need some results about the first derived functor of the inverse limit functor on countable inverse systems, and also on Mittag-Leffler and T-nilpotent countable inverse system of abelian groups. Let us start with a definition:

Definition 7.3. Given a countable inverse system

$$\cdots \to H_{n+1} \overset{h_n}{\to} H_n \to \cdots \to H_2 \overset{h_1}{\to} H_1 \overset{h_0}{\to} H_0$$

of abelian groups, we say that the system is *Mittag-Leffler* if for each n the descending chain

$$H_n \supseteq h_n(H_{n+1}) \supseteq \cdots \supseteq h_n h_{n+1} \cdots h_{k-1}(H_k) \supseteq \cdots$$

is stationary. Moreover, we say that the inverse system is *T-nilpotent* if for each n there exists $k > n$ such that the composition $H_k \to H_n$ is zero.

Let us also recall that the inverse limit $\varprojlim H_n$ of such an inverse system $(H_n, h_n \mid n < \omega)$, and the *first derived functor of the inverse limit*, $\varprojlim^1 H_n$, can be computed using the exact sequence

$$0 \to \varprojlim H_n \to \prod H_n \overset{\Delta}{\to} \prod H_n \to \varprojlim^1 H_n \to 0$$

where $\Delta((x_n)_{n<\omega}) = (x_n - h_n(x_{n+1}))_{n<\omega}$. The first derived functor or $\varprojlim$ is closely related to the fact that inverse limits are not exact—they are only left exact in general. The notions of the inverse limit and its first derived functor are closely related to the concepts from Definition 7.3:

Proposition 7.4. *Let $(H_n, h_n \mid n < \omega)$ be a countable inverse system of abelian groups. Then the following hold:*

(1) *If (H_n, h_n) is Mittag-Leffler, then $\varprojlim^1 H_n = 0$.*
(2) *(H_n, h_n) is Mittag-Leffler if and only if $\varprojlim^1 H_n^{(\omega)} = 0$.*
(3) *(H_n, h_n) is T-nilpotent if and only if it is Mittag-Leffler and $\varprojlim H_n = 0$.*

Proof. (1) is proved in [26, Proposition 3.5.7], (2) in [1, Theorem 1.3], and (3) in [20, Lemma 4.5]. □

Now we are ready to prove Theorem 7.1:

Proof of Theorem 7.1. Suppose we are given a class $\mathcal{D}$ of cotorsion R-modules which is closed under arbitrary direct sums. Let $\mathcal{S}$ be a representative set of countably generated and bounded complexes from $\mathbf{C}^{\leq 0}(\text{Flat-}R) \cap {}^{\perp}\mathcal{D}$.

It is straightforward to see, using Lazard's theorem, that for each $F \in \mathcal{S}$, there is a countable direct system

$$F_0 \xrightarrow{f_0} F_1 \xrightarrow{f_1} F_2 \xrightarrow{f_2} F_3 \longrightarrow \cdots$$

of complexes with finitely generated projective components, concentrated in the same degrees as F, and such that $F = \varinjlim F_i$. This gives a short exact sequence $0 \to \bigoplus F_n \to \bigoplus F_n \to F \to 0$ in $\mathbf{C}(\text{Mod-}R)$, and so the corresponding triangle

$$\bigoplus F_n \xrightarrow{\ \delta\ } \bigoplus F_n \longrightarrow F \longrightarrow \bigoplus F_i[1] \tag{v}$$

in $\mathbf{D}(\text{Mod-}R)$. When we apply the functor $\operatorname{Ext}^i_R(-,C)$ on the morphism δ for some $i \geq 0$ and $C \in \mathcal{D}$, it is easy to see that we get the exact sequence:

$$0 \to \varprojlim \operatorname{Ext}^i_R(F_n, C) \to \prod \operatorname{Ext}^i_R(F_n, C) \to \prod \operatorname{Ext}^i_R(F_n, C) \to \varprojlim{}^1 \operatorname{Ext}^i_R(F_n, C) \to 0$$

Note also that since all F_n are bounded complexes of finitely generated projective modules, the functors $\operatorname{Ext}^i_R(F_n,-) \cong \operatorname{Hom}_{\mathbf{K}(\text{Mod-}R)}(F_n[-i],-) : \mathbf{C}(\text{Mod-}R) \to \text{Ab}$ commute with direct limits.

Now we define the class $\overline{\mathcal{D}}$. An R-module C belongs to $\overline{\mathcal{D}}$ if:

(1) The inverse system $(\operatorname{Ext}^0_R(F_n, C), \operatorname{Ext}^0_R(f_n, C))$ is Mittag-Leffler for each $C \in \mathcal{D}$;
(2) The inverse system $(\operatorname{Ext}^i_R(F_n, C), \operatorname{Ext}^i_R(f_n, C))$ is T-nilpotent each $C \in \mathcal{D}$ and $i \geq 1$;

It is straightforward to see that $\overline{\mathcal{D}}$ is definable. We will next prove that $\mathcal{D} \subseteq \overline{\mathcal{D}}$. If $C \in \mathcal{D}$ and $F = \varinjlim F_n \in \mathcal{S}$, we can apply $\operatorname{Ext}^i_R(-, C^{(\omega)})$ on triangle (v) to get:

$$\begin{aligned} &\prod \operatorname{Ext}^0_R(F_n, C)^{(\omega)} \to \prod \operatorname{Ext}^0_R(F_n, C)^{(\omega)} \to \\ &\quad \to \operatorname{Ext}^1_R(F, C^{(\omega)}) \to \prod \operatorname{Ext}^1_R(F_n, C)^{(\omega)} \to \prod \operatorname{Ext}^1_R(F_n, C)^{(\omega)} \to \dots \\ &\quad \dots \to \operatorname{Ext}^i_R(F, C^{(\omega)}) \to \prod \operatorname{Ext}^i_R(F_n, C)^{(\omega)} \to \prod \operatorname{Ext}^i_R(F_n, C)^{(\omega)} \to \dots \end{aligned}$$

Since $\operatorname{Ext}^i_R(F, C^{(\omega)}) = 0$ for each $i \geq 1$, it follows by Proposition 7.4(2) that $(\operatorname{Ext}^i_R(F_n, C), \operatorname{Ext}^1_R(f_n, C))$ is Mittag-Leffler for each $i \geq 0$. Furthermore, if we apply $\operatorname{Ext}^i_R(-, C)$ on (v), we see that for each $i \geq 1$ the morphism $\operatorname{Ext}^i_R(\delta, C)$ is an isomorphism, so $\varprojlim \operatorname{Ext}^i_R(F_n, C) = 0$, and consequently $(\operatorname{Ext}^i_R(F_n, C), \operatorname{Ext}^1_R(f_n, C))$ is T-nilpotent by Proposition 7.4(3). This shows that $C \in \overline{\mathcal{D}}$.

Finally, let us show that each $C \in \overline{\mathcal{D}}$ is cotorsion. Indeed, if we have such a C, then by Proposition 7.4 we deduce that for each $F = \varinjlim F_n \in \mathcal{S}$:

(1) $\operatorname{Ext}^0_R(\delta, C)$ is an epimorphism; and
(2) $\operatorname{Ext}^i_R(\delta, C)$ is an isomorphism for each $i \geq 1$.

Hence, if we apply $\operatorname{Ext}^i_R(-, C)$ on triangle (v), we get

$$\begin{aligned} &\prod \operatorname{Ext}^0_R(F_n, C) \twoheadrightarrow \prod \operatorname{Ext}^0_R(F_n, C) \xrightarrow{0} \\ &\quad \xrightarrow{0} \operatorname{Ext}^1_R(F, C) \xrightarrow{0} \prod \operatorname{Ext}^1_R(F_n, C) \xrightarrow{\sim} \prod \operatorname{Ext}^1_R(F_n, C) \xrightarrow{0} \dots \\ &\quad \overset{0}{\dots} \to \operatorname{Ext}^i_R(F, C) \xrightarrow{0} \prod \operatorname{Ext}^i_R(F_n, C) \xrightarrow{\sim} \prod \operatorname{Ext}^i_R(F_n, C) \xrightarrow{0} \dots \end{aligned}$$

In particular, $\operatorname{Ext}^i_R(F, C) = 0$ for each $F \in \mathcal{S}$ and $i \geq 1$.

Let now G be any flat module and $P \in \mathbf{C}^{\leq 0}(\text{Proj-}R)$ a projective resolution of G. Then P has a filtration $(P_\alpha \mid \alpha \leq \tau)$ such that $P_{\alpha+1}/P_\alpha$ is concealed bounded and belongs to $\mathbf{C}^{\leq 0}(\text{Proj-}R) \cap {}^{\perp}\mathcal{D}$ for each $\alpha < \tau$.

In particular, each $P_{\alpha+1}/P_\alpha$ is isomorphic in $\mathbf{D}(\text{Mod-}R)$ to some $F \in \mathcal{S}$. Therefore, $\operatorname{Ext}^i_R(P_{\alpha+1}/P_\alpha, C) = 0$ for each $\alpha < \tau$ and $i \geq 1$. Applying Proposition 3.4, we get:

$$0 = \operatorname{Ext}^i_R(P, C) = \operatorname{Ext}^i_R(G, C) \text{ for each } i \geq 1.$$

Hence C has no extensions by flat modules, which by definition means that C is cotorsion. $\square$

References

[1] L. Angeleri Hügel, S. Bazzoni, and D. Herbera. A solution to the Baer splitting problem. *Trans. Amer. Math. Soc.*, 360(5):2409–2421, 2008.

[2] S. Bazzoni and J. Šťovíček. Sigma-cotorsion modules over valuation domains. *Forum Math.*, 21(5):893–920, 2009.

[3] M. Bökstedt and A. Neeman. Homotopy limits in triangulated categories. *Compositio Math.*, 86(2):209–234, 1993.

[4] W. Crawley-Boevey. Locally finitely presented additive categories. *Comm. Algebra*, 22(5):1641–1674, 1994.

[5] W. Crawley-Boevey. Infinite-dimensional modules in the representation theory of finite-dimensional algebras. In *Algebras and modules, I (Trondheim, 1996)*, volume 23 of *CMS Conf. Proc.*, pages 29–54. Amer. Math. Soc., Providence, RI, 1998.

[6] P. C. Eklof. Shelah's singular compactness theorem. *Publ. Mat.*, 52(1):3–18, 2008.

[7] P. C. Eklof and A. H. Mekler. *Almost free modules*, volume 46 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1990. Set-theoretic methods.

[8] R. Göbel and J. Trlifaj. *Approximations and endomorphism algebras of modules*, volume 41 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter GmbH & Co. KG, Berlin, 2006.

[9] P. A. Guil Asensio and I. Herzog. Left cotorsion rings. *Bull. London Math. Soc.*, 36(3):303–309, 2004.

[10] P. A. Guil Asensio and I. Herzog. Sigma-cotorsion rings. *Adv. Math.*, 191(1):11–28, 2005.

[11] P. A. Guil Asensio and I. Herzog. Pure-injectivity in the category of flat modules. In *Algebra and its applications*, volume 419 of *Contemp. Math.*, pages 155–166. Amer. Math. Soc., Providence, RI, 2006.

[12] P. A. Guil Asensio and I. Herzog. Indecomposable flat cotorsion modules. *J. Lond. Math. Soc. (2)*, 76(3):797–811, 2007.

[13] P. A. Guil Asensio and I. Herzog. Model-theoretic aspects of Σ-cotorsion modules. *Ann. Pure Appl. Logic*, 146(1):1–12, 2007.

[14] D. Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*, volume 119 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.

[15] R. Hartshorne. *Residues and duality*. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin, 1966.

[16] I. Herzog. Pure-injective envelopes. *J. Algebra Appl.*, 2(4):397–402, 2003.

[17] I. Kaplansky. Projective modules. *Ann. of Math (2)*, 68:372–377, 1958.

[18] B. Keller. Chain complexes and stable categories. *Manuscripta Math.*, 67(4):379–417, 1990.

[19] M. Prest. *Model theory and modules*, volume 130 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.

[20] J. Šaroch and J. Šťovíček. The countable telescope conjecture for module categories. *Adv. Math.*, 219(3):1002–1036, 2008.

[21] N. Spaltenstein. Resolutions of unbounded complexes. *Compositio Math.*, 65(2):121–154, 1988.

[22] J. Šťovíček and J. Trlifaj. All tilting modules are of countable type. *Bull. Lond. Math. Soc.*, 39(1):121–132, 2007.

[23] J. Šťovíček and J. Trlifaj. Generalized Hill lemma, Kaplansky theorem for cotorsion pairs and some applications. *Rocky Mountain J. Math.*, 39(1):305–324, 2009.

[24] W. Szmielew. Elementary properties of Abelian groups. *Fund. Math.*, 41:203–271, 1955.

[25] J. Trlifaj. Roots of Ext, their abstract elementary classes, deconstruction, and applications. In *Proceedings of the International Conference on Modules and Representation Theory*, pages 195–209. Presa Univ. Clujeană, Cluj-Napoca, 2009.

[26] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[27] M. Ziegler. Model theory of modules. *Ann. Pure Appl. Logic*, 26(2):149–213, 1984.

Charles University in Prague, Faculty of Mathematics and Physics, Department of Algebra, Sokolovska 83, 186 75 Praha 8, Czech Republic

E-mail address: `stovicek@karlin.mff.cuni.cz`